

# PARABOLIC AND NEAR-PARABOLIC RENORMALIZATION FOR LOCAL DEGREE THREE

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**ABSTRACT.** The theory of parabolic and near-parabolic renormalization developed by Inou and Shishikura behaves extremely useful in recent years. It leads to several progresses on the dynamics of quadratic polynomials. In this paper, we give an invariant class and extend the results of Inou and Shishikura to local degree three.

## 1. INTRODUCTION

Let  $f(z)$  be a holomorphic function defined in a neighborhood of  $z_0 \in \mathbb{C}$  and suppose that  $z_0$  is a fixed point of  $f$ . The number  $f'(z_0)$  is called the *multiplier* of  $z_0$ . If  $f'(z_0)$  is a root of the unity, then  $z_0$  is called a *parabolic fixed point* of  $f$ . In particular,  $z_0$  is called *1-parabolic* if  $f'(z_0) = 1$  and it is called *non-degenerate* if  $f''(z_0) \neq 0$ . The *parabolic basin* of  $z_0$  is defined as

$$\text{Basin}(z_0) = \{z : \{f^{on}\}_{n=0}^{\infty} \text{ converges uniformly to } z_0 \text{ in a neighborhood of } z\}.$$

The dynamical behavior in the parabolic basin is simple but it may become complicated once a perturbation on the map  $f$  has been made. The main tools to analyze such complicated phenomena are Fatou coordinates and horn maps, which were developed by Douady-Hubbard [DH84-85, DH85], Lavaurs [Lav89] and Inou-Shishikura [IS08].

The definitions of parabolic and near-parabolic renormalization were introduced in [Shi98], where the Hausdorff dimension of the Mandelbrot set is equal to two was proved. Near-parabolic renormalization is also called *cylinder renormalization*, which was introduced by Yampolsky in the study of analytic circle homeomorphisms with a critical point [Yam03]. In [Shi98], an invariant class under the parabolic renormalization was introduced. However, in order to iterate the near-parabolic renormalization infinitely many times, the class defined in [Shi98] cannot serve for the purpose anymore. Later, a more larger class was introduced in [IS08].

There are several applications by using Inou-Shishikura's invariant class defined in [IS08]. The first remarkable application is that Buff and Chéritat used it as one of the main tools to prove the existence of quadratic polynomials with positive area [BC12]. Recently, Cheraghi and his coauthors have found several other important applications. In [Che10] and [Che13], Cheraghi developed several elaborate analytic techniques based on the Inou-Shishikura's result. The tools in [Che10] and [Che13] have led to some of the recent major progresses on the dynamics of quadratic polynomials. For example, the Feigenbaum Julia sets with positive area (which is very different from the examples in [BC12]) have been found in [AL15], the Marmi-Moussa-Yoccoz conjecture for rotation numbers of high type has been proved in [CC15], the local connectivity of the Mandelbrot set at some infinitely satellite renormalizable points was proved in [CS15], some statistical properties of the dynamics of quadratic polynomials was depicted in [AC12] etc.

In Inou-Shishikura's class, each map has only one critical point and only one critical value. Moreover, the critical point is simple (i.e. with local degree two). Recently, Chéritat

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extended the parabolic and near-parabolic theory to all finite degrees with several critical points and with only one critical value in [Ché14]. However, in some sense, the classes found in [Ché14] are smaller than Inou-Shishikura's class. Specifically, in the case of degree two, the "structure" of the Inou-Shishikura's class is the restriction of the "structure" of Chéritat's class. For more detailed description on the "structure", see §4 or [Ché14, §1]. Therefore, a natural question is to extend Inou-Shishikura's class to higher degrees and with only one critical point, i.e. such that the classes in the higher degrees are as big as Inou-Shishikura's. Our main goal in this paper is to do this for the cubic case.

We will adapt the method of Inou-Shishikura and imitate the framework of their paper since we also need to do lots of calculations and numerical estimations. We need to check a number of inequalities, and about 70 of them were verified by the help of the computer. These inequalities are about the maximums or minimums of elementary functions on closed intervals, or evaluated at explicit values. The software we chose in this paper was Mathematica 9.0. See [Yan15] for actual calculations. Although our calculation in this paper only works for the cubic case, the reader also has hope to work for higher degrees in the similar way, especially the idea may work for higher *odd* degrees.

The most difficult and important point among the extension of Inou-Shishikura's result to higher degrees is to define new classes (especially the choice of the domains of definitions). Therefore, before stating the main results, we first introduce a class as in the following:

**Definition.** Let  $P(z) = z(1 + \frac{2\sqrt{5}}{3}z + z^2)^2$ . Then  $P : \mathbb{C} \rightarrow \mathbb{C}$  has a parabolic fixed point at 0 and four critical points  $-\frac{1}{\sqrt{5}}$  (with local degree *three*) and  $c_{\pm} = -\frac{\sqrt{5}}{3} \pm \frac{2}{3}i$  with  $P(c_+) = P(c_-) = 0$  and  $P(-\frac{1}{\sqrt{5}}) = -\frac{64}{225\sqrt{5}}$ . Let  $V$  be a domain of  $\mathbb{C}$  containing 0 and define<sup>1</sup>

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \rightarrow \mathbb{C} \mid \begin{array}{l} \varphi : V \rightarrow \mathbb{C} \text{ is univalent, } \varphi(0) = 0, \varphi'(0) = 1 \\ \text{and } \varphi \text{ has a quasiconformal extension to } \mathbb{C} \end{array} \right\}.$$

If  $f \in \mathcal{F}_1$ , then 0 is a 1-parabolic fixed point of  $f$ . If  $-\frac{1}{\sqrt{5}} \in V$ , then  $cp_f := \varphi(-\frac{1}{\sqrt{5}})$  is a critical point of  $f$  with local degree *three* and  $-\frac{64}{225\sqrt{5}}$  is a critical value of  $f$ .

We now state the main results of this paper. They are almost copies of the three main theorems and corresponding corollaries in [IS08, §4]. For the definitions of parabolic renormalization  $\mathcal{R}_0$ , near-parabolic renormalization  $\mathcal{R}$  and the renormalization  $\mathcal{R}_\alpha$  in the fiber direction, see §2. The reader is also strongly recommended to refer [Shi98], [Shi00] and [IS08] for more detailed and comprehensive description.

**Main Theorem** (Invariance of  $\mathcal{F}_1$ ). *There exist a Jordan domain  $V$  containing 0 and  $-\frac{1}{\sqrt{5}}$  with a smooth boundary and an open set  $V'$  containing  $\bar{V}$  such that  $\mathcal{F}_1$  satisfies the following:*

- (a)  $\forall f \in \mathcal{F}_1$ ,  $f''(0) \neq 0$  (specifically,  $|f''(0) - 6.74| \leq 1.93$ ) and the critical point  $cp_f$  is contained in  $\text{Basin}(0)$ .
- (b)  $\mathcal{R}_0(\mathcal{F}_1) \subset \mathcal{F}_1$ . Namely, for  $f \in \mathcal{F}_1$ , the parabolic renormalization  $\mathcal{R}_0 f$  is well-defined so that  $\mathcal{R}_0 f = P \circ \psi^{-1} \in \mathcal{F}_1$ . Moreover,  $\psi$  extends to a univalent function from  $V'$  to  $\mathbb{C}$ .
- (c)  $\mathcal{R}_0$  is holomorphic in the following sense: Suppose that a family  $f_\lambda = P \circ \varphi_\lambda^{-1}$  is given by a holomorphic function  $\varphi_\lambda(z)$  in two variables  $(\lambda, z) \in \Lambda \times V$ , where  $\Lambda$  is a complex manifold. Then the parabolic renormalization can be written as  $\mathcal{R}_0 f_\lambda = P \circ \psi_\lambda^{-1}$  with  $\psi_\lambda(z)$  holomorphic in  $(\lambda, z) \in \Lambda \times V'$ .

<sup>1</sup>We need to find a polynomial with a 1-parabolic fixed point at 0 whose immediate parabolic basin contains exactly one critical point with local degree three and all other critical points are mapped onto the origin. It is easy to prove that any real polynomial does not satisfy this property (A polynomial is called *real* if its coefficients and critical points are all real). Note that the polynomial  $P$  we introduce here is not a real polynomial since not all critical points of  $P$  lie on the real axis. There maybe exist several other choices. But the  $P$  we choose here is monic and has symmetric dynamical properties about the real axis.

This theorem is central in this paper and the explicit definition of  $V$  and  $V'$  will be given in §3.1 (see also Figure 2). The requirement that  $\bar{V} \subset V'$  is important. This means that the new definition of domain for  $\psi$  is strictly larger than that of the original  $\varphi$ . Comparing the above theorem with the Main Theorem 1 in [IS08], one may find that the statement on another class  $\mathcal{F}_0$  was lost. In fact, we have a corresponding statement but there exist some differences (see Proposition 3.3).

**Theorem 1.1** (Contraction). *There exists a one to one correspondence between  $\mathcal{F}_1$  and the Teichmüller space of  $\mathbb{C} \setminus \bar{V}$ . Let  $d(\cdot, \cdot)$  be the distance on  $\mathcal{F}_1$  induced from the Teichmüller distance, which is complete. Then  $\mathcal{R}_0$  is a uniform contraction:*

$$d(\mathcal{R}_0(f), \mathcal{R}_0(g)) \leq \lambda d(f, g) \text{ for } f, g \in \mathcal{F}_1,$$

where  $\lambda = e^{-2\pi \operatorname{mod}(V' \setminus \bar{V})} < 1$ . The convergence with respect to  $d$  implies the uniform convergence on compact sets (but not vice versa).

Theorem 1.1 has been proved by Inou and Shishikura in [IS08, §6] for their class (Main Theorem 2 in their paper). However, this theorem and its proof are “almost” unrelated to the specific class  $\mathcal{F}_1$ . The unique essential requirement in Theorem 1.1 is  $\bar{V} \subset V'$ , which we have proved in the Main Theorem (The requirement that the normalized univalent map  $\varphi : V \rightarrow \mathbb{C}$  has a quasiconformal extension to  $\mathbb{C}$  in the proof is included in the definition of  $\mathcal{F}_1$ ). Therefore, this theorem can be seen as a direct corollary of the Main Theorem (by following Inou-Shishikura’s proof).

**Definition.** For  $\alpha_* > 0$ , denote  $\mathcal{I}(\alpha_*) = (-\alpha_*, \alpha_*) \setminus \{0\}$  and

$$e^{2\pi i \mathcal{I}(\alpha_*)} \times \mathcal{F}_1 = \{e^{2\pi i \alpha} h(z) \mid \alpha \in \mathcal{I}(\alpha_*) \text{ and } h \in \mathcal{F}_1\}.$$

The distance on this space is defined as  $d(f, g) = d(\frac{f}{f'(0)}, \frac{g}{g'(0)}) + |f'(0) - g'(0)|$ , where  $d$  on the right hand of the equation is the distance appeared in Theorem 1.1.

Let  $\alpha \in (0, 1) \setminus \mathbb{Q}$  be an irrational number. Then  $\alpha$  has a continued fraction expansion  $[a_1, a_2, \dots, a_n, \dots] := 1/(a_1 + 1/(a_2 + 1/(\dots)))$ . For an positive integer  $N$ , let  $\operatorname{Irrat}_{\geq N}$  be the set of irrational number such that the continued fraction expansion has coefficients  $a_n \geq N$ .

**Theorem 1.2** (Invariance of  $\mathcal{F}_1$  under  $\mathcal{R}_\alpha$  and hyperbolicity). *There exists  $\alpha_* > 0$  such that if  $\alpha \in \mathbb{C}$ ,  $|\arg \alpha| < \pi/4$  (or  $|\arg(-\alpha)| < \pi/4$ ) and  $0 < |\alpha| \leq \alpha_*$ , then the fiber renormalization  $\mathcal{R}_\alpha$  can be defined in  $\mathcal{F}_1$  such that (b) and (c) of the Main Theorem hold for  $\mathcal{R}_\alpha$ . Moreover  $\mathcal{R}_\alpha$  is a contraction as in Theorem 1.1 with the same  $\lambda$ . Hence the near-parabolic renormalization  $\mathcal{R}$  is hyperbolic in  $e^{2\pi i \mathcal{I}(\alpha_*)} \times \mathcal{F}_1$ .*

*In particular, there exists an integer  $N \geq 2$  for which the following holds: If  $f(z) = e^{2\pi i \alpha} h(z)$  with  $h \in \mathcal{F}_1$  and  $\alpha \in \operatorname{Irrat}_{\geq N}$ , then an infinite sequence of near-parabolic renormalizations beginning with  $f$  can be defined and they all belong to  $e^{2\pi i \mathcal{I}(\alpha_*)} \times \mathcal{F}_1$ . If  $g(z)$  is another map of the same type with the same  $\alpha$ , then  $d(\mathcal{R}^n f, \mathcal{R}^n g) \rightarrow 0$  as  $n \rightarrow \infty$  exponentially fast.*

**Corollary 1.3.** *There exists an  $N$  (may be larger than the one in Theorem 1.2) such that if  $f(z) = e^{2\pi i \alpha} h(z)$  with  $h \in \mathcal{F}_1$  and  $\alpha \in \operatorname{Irrat}_{\geq N}$ , then the critical orbit of  $f$  stays in the domain of definition of  $f$  and can be iterated infinitely many times. Moreover there exists an infinite sequence of periodic orbits to which the critical orbit does not accumulate.*

*The same conclusion holds for  $f(z) = e^{2\pi i \alpha}(z + \sqrt{3}z^2 + z^3)$  provided that  $\alpha \in \operatorname{Irrat}_{\geq N}$  and  $\alpha$  itself is sufficiently small. Hence the critical orbit is not dense in the Julia set of  $f$ .*

For Theorem 1.2, the numbers  $\alpha_*$  and  $N$  can be obtained by a continuity argument as in the proof of the Main Theorem 2 in [IS08, §7]. The infinite sequence of near-parabolic renormalizations beginning with  $f$  in this theorem is defined in §2. For Corollary 1.3, note

that  $f(z) = e^{2\pi i\alpha}(z + \sqrt{3}z^2 + z^3)$  is a cubic polynomial and  $z = -1/\sqrt{3}$  is a critical point of  $f$  with local degree three. Although  $z \mapsto z + \sqrt{3}z^2 + z^3$  is not contained in  $\mathcal{F}_1$ , but the first parabolic renormalization  $\mathcal{R}_0(z + \sqrt{3}z^2 + z^3)$  is. In particular,  $\mathcal{R}_0(z + \sqrt{3}z^2 + z^3)$  is contained in  $\mathcal{F}_2^P$ , which is a subclass of  $\mathcal{F}_1$  (see Proposition 3.3). Then, for sufficiently small  $\alpha$ , one has  $\mathcal{R}_\alpha(z + \sqrt{3}z^2 + z^3) \in \mathcal{F}_1$  and the proof of Corollary 1.3 can be completed as in [IS08, §7].

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**Notations.** We use  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$  and  $\mathbb{C}$  to denote the set of all natural numbers, integers, rational numbers, real numbers and complex numbers, respectively. The Riemann sphere, the upper (below) half plane and the unit disk are denoted by  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ ,  $\mathbb{H}^\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$  and  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  respectively. A disk is denoted by  $\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$  and  $\overline{\mathbb{D}(a, r)}$  is its closure. We use  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ,  $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$  and  $\mathbb{R}_\pm$  to denote the punched complex plane, punched unit disk and the positive (negative) real line respectively. We use  $d_X(\cdot, \cdot)$  to denote the Poincaré distance on the hyperbolic Riemann surface  $X$  and  $\mathbb{D}_X(a, r) = \{z \in X : d_X(z, a) < r\}$  denotes the hyperbolic disk in  $X$ . For a function  $f(z)$ , we denote  $f^*(z) = \overline{f(\overline{z})}$ . For a complex number  $z \neq 0$ ,  $\arg z$  and  $\log z$  denote a suitably chosen branch of the argument and logarithm respectively. For a complex number  $a \neq 0$  and a set  $Z \subset \mathbb{C}$ , we denote  $aZ = \{az : z \in Z\}$ .

## 2. PARABOLIC FIXED POINTS AND THEIR BIFURCATION

In this section, we review the definitions and theories of Fatou coordinates, horn maps, parabolic renormalization and near-parabolic renormalization etc. For their proofs and more details, see [Shi98, Shi00] and [IS08]. The reader who is familiar with the theories and notations can skip this section safely.

**2.1. Fatou coordinates and horn maps.** Let  $f(z) = z + a_2z^2 + \mathcal{O}(z^3)$  be a holomorphic function with  $a_2 \neq 0$ . Then  $f$  has a non-degenerated 1-parabolic fixed point at the origin. After a coordinate change  $w = -\frac{1}{a_2z}$ , the parabolic fixed point 0 moves to  $\infty$  and the dynamics of  $f$  in this new coordinate is  $F(w) = w + 1 + \frac{b_1}{w} + \mathcal{O}(\frac{1}{w^2})$  near  $\infty$ , where  $b_1 \in \mathbb{C}$  is a constant<sup>2</sup> depending on  $f$ .

**Theorem 2.1** (and the definitions of Fatou coordinates and horn map). *For a sufficiently large  $L > 0$ , there are univalent maps  $\Phi_{attr} = \Phi_{attr, F} : \{w : \operatorname{Re} w - L > -|\operatorname{Im} w|\} \rightarrow \mathbb{C}$  and  $\Phi_{rep} = \Phi_{rep, F} : \{w : \operatorname{Re} w + L < |\operatorname{Im} w|\} \rightarrow \mathbb{C}$  such that*

$$\Phi_s(F(w)) = \Phi_s(w) + 1 \quad (s = attr, rep)$$

*holds in the region where both sides are defined. These two maps  $\Phi_{attr}$  and  $\Phi_{rep}$  are unique up to an additive constant. They are called attracting and repelling Fatou coordinates respectively.*

*In the region  $V_\pm = \{w : \pm \operatorname{Im} w > |\operatorname{Re} w| + L\}$ , both Fatou coordinates are defined. The horn map  $E_F$  on  $\Phi_{rep}(V_\pm)$  is defined as*

$$E_F = \Phi_{attr} \circ \Phi_{rep}^{-1}.$$

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<sup>2</sup>If  $f(z) = z + a_2z^2 + a_3z^3 + \text{higher order terms}$  with  $a_2 \neq 0$ , then an easy calculation shows that the constant  $b_1 = 1 - \frac{a_3}{a_2^2}$ , which is called the *iterative residue* of  $f$ .

There exists a more larger  $L' > 0$  such that  $E_F$  extends holomorphically to  $\{z : |\operatorname{Im} z| \geq L'\}$  and satisfies  $E_F(z+1) = E_F(z) + 1$ . Moreover, there are constants  $c_{upper}$  and  $c_{lower}$  such that

$$E_F(z) - z \rightarrow c_{upper} \text{ as } \operatorname{Im} z \rightarrow +\infty \text{ and } E_F(z) - z \rightarrow c_{lower} \text{ as } \operatorname{Im} z \rightarrow -\infty,$$

and  $c_{lower} - c_{upper} = 2\pi i b_1$ .

Since the Fatou coordinates are unique up to an additive constant, it is convenient to make a normalization for them. In this paper, the attracting Fatou coordinate is *normalized* by  $\Phi_{attr}(cp) = 0$  for a special critical point  $cp$ . For the repelling Fatou coordinate, we normalize it such that  $c_{upper} = 0$ , i.e.

$$E_F(z) = z + o(1) \text{ as } \operatorname{Im} z \rightarrow +\infty.$$

We need to consider a sequence of functions converging to a limiting function and sometimes we will use the term that a function is “close” to another. Hence the definition of a neighborhood of a function is needed.

**Definition** (Neighborhood of a function). For a give function  $f$ , its domain of definition is denoted by  $\operatorname{Dom}(f)$ . A *neighborhood* of  $f$  is

$$\mathcal{N} = \mathcal{N}(f; K, \varepsilon) = \{g : \operatorname{Dom}(g) \rightarrow \widehat{\mathbb{C}} \mid K \subset \operatorname{Dom}(g) \text{ and } \sup_{z \in K} d_{\widehat{\mathbb{C}}}(g(z), f(z)) < \varepsilon\},$$

where  $d_{\widehat{\mathbb{C}}}$  denotes the spherical distance,  $K$  is a compact set contained in  $\operatorname{Dom}(f)$  and  $\varepsilon > 0$ . A sequence  $\{f_n\}$  is called *converges to  $f$  uniformly on compact sets* if for any neighborhood  $\mathcal{N}$  of  $f$ , there exists  $n_0 > 0$  such that  $f_n \in \mathcal{N}$  for all  $n \geq n_0$ .

The horn map constructed by means of  $f \rightsquigarrow E_f$  is continuous and holomorphic in some sense. For the continuity, roughly speaking, if  $f$  is a function with a non-degenerate 1-parabolic fixed point at 0 and suppose that  $\mathcal{N}$  is a neighborhood of the horn map  $E_f$ , there exists a neighborhood  $\mathcal{N}'$  of  $f$  such that if  $g \in \mathcal{N}'$  with a non-degenerate 1-parabolic fixed point at 0, then  $E_g \in \mathcal{N}$ . See [IS08, Theorem 1.3] for the statement and [Shi00] for the proof.

**2.2. Parabolic renormalization and near-parabolic renormalization.** Let  $f_0$  be a holomorphic function with a non-degenerate 1-parabolic fixed point at  $z = 0$ . We consider the perturbation  $f$  which is close to  $f_0$  in a neighborhood of 0. Note that 0 is a solution of  $f(z) = z$  with multiplicity 2. Then  $f$  has two fixed points (counted with multiplicity) near the origin. After a coordinate transformation, we suppose that one of the fixed point of  $f$  is still at 0 and with multiplier  $e^{2\pi i \alpha}$ . The perturbation is always assumed to be in the direction  $|\arg \alpha| < \frac{\pi}{4}$  or  $|\arg(-\alpha)| < \frac{\pi}{4}$  since the complicated and interesting bifurcation phenomena appears only in these cases. The latter case can be reduced to the former by a complex conjugacy, i.e. one can consider  $f^*(z) = \overline{f(\bar{z})}$ . Therefore, we consider the perturbation  $f$  having the form:

$$f(z) = e^{2\pi i \alpha} z + \mathcal{O}(z^2) \text{ with small } \alpha \text{ and } |\arg \alpha| < \frac{\pi}{4}. \quad (2.1)$$

Let  $\sigma(f)$  be the other fixed point of  $f$  near 0. The proof of following theorem can be found in [Shi98] and [Shi00] and the statement can be found in [IS08].

**Theorem 2.2** (and the definition of Fatou coordinates and horn map for the perturbed map). *Suppose that  $f_0$  has a non-degenerate 1-parabolic fixed point at  $z = 0$ . Then there exists a neighborhood  $\mathcal{N} = \mathcal{N}(f_0; K, \varepsilon)$  ( $0$  should be contained in  $\operatorname{int} K$ ) such that if  $f \in \mathcal{N}$  and  $f$  satisfies (2.1), then the fundamental regions  $S_{attr,f}$  and  $S_{rep,f}$  exist. The Fatou coordinates  $\Phi_{attr,f}$  and  $\Phi_{rep,f}$  are defined in a neighborhood of  $\overline{S}_{attr,f} \setminus \{0, \sigma(f)\}$  and  $\overline{S}_{rep,f} \setminus \{0, \sigma(f)\}$ . The horn map  $E_f$  can be similarly defined as in Theorem 2.1. After a suitable normalization,  $\Phi_{attr,f}$ ,  $\Phi_{rep,f}$  and  $E_f$  depend continuously and holomorphically on  $f$ .*

Based on these preparations, we can now define parabolic renormalization and near-parabolic renormalization.

**Definition** (Parabolic renormalization). Denote  $\text{Exp}^\sharp(z) = e^{2\pi iz}$  and  $\text{Exp}^\flat(z) = e^{-2\pi iz}$ . They both map  $\mathbb{C}/\mathbb{Z}$  isomorphically onto  $\mathbb{C}^*$ . Suppose that  $f$  has a non-degenerate 1-parabolic fixed point at 0. Its *parabolic renormalization* is defined as

$$\mathcal{R}_0 f = \mathcal{R}_0^\sharp f = \text{Exp}^\sharp \circ E_f \circ (\text{Exp}^\sharp)^{-1},$$

where  $E_f$  is the horn map of  $f$ , defined in Theorem 2.1 and normalized as  $E_f(z) = z + o(1)$  as  $\text{Im } z \rightarrow +\infty$ . Then  $\mathcal{R}_0 f$  extends holomorphically to 0 and  $\mathcal{R}_0 f(0) = 0$ ,  $(\mathcal{R}_0 f)'(0) = 1$ . So  $\mathcal{R}_0 f$  has again a 1-parabolic fixed point at 0.

The parabolic renormalization for lower end is defined as

$$\mathcal{R}_0^\flat f = c \text{Exp}^\flat \circ E_f \circ (\text{Exp}^\flat)^{-1},$$

where  $c \in \mathbb{C}^*$  is chosen such that  $(\mathcal{R}_0^\flat f)'(0) = 1$ .

**Definition** (Near-parabolic renormalization). Suppose that  $f(z) = e^{2\pi i \alpha} z + \mathcal{O}(z^2)$  with  $\alpha \neq 0$  and has fundamental domains stated in Theorem 2.2 (The number  $\alpha$  is supposed to be small and  $|\arg \alpha| < \frac{\pi}{4}$ ). Let

$$\chi_f(z) = z - \frac{1}{\alpha(f)} \text{ on } \mathbb{C}/\mathbb{Z}.$$

The *near-parabolic renormalization* (or *cylinder renormalization*) of  $f$  is defined by

$$\mathcal{R}f = \mathcal{R}^\sharp f = \text{Exp}^\sharp \circ \chi_f \circ E_f \circ (\text{Exp}^\sharp)^{-1}.$$

Then  $\mathcal{R}f$  extends holomorphically to 0 and  $\mathcal{R}f(0) = 0$ ,  $(\mathcal{R}f)'(0) = e^{-2\pi i \frac{1}{\alpha}}$ .

For  $\alpha$  with  $|\arg(-\alpha)| < \frac{\pi}{4}$ , the above construction can be applied to  $f^*(z) = \overline{f(\bar{z})}$  and define  $\mathcal{R}f = \mathcal{R}^\sharp f = \text{Exp}^\sharp \circ \chi_{f^*} \circ E_{f^*} \circ (\text{Exp}^\sharp)^{-1}$ . For the lower end, define  $\mathcal{R}^\flat f$  replacing  $\text{Exp}^\sharp$  by  $\text{Exp}^\flat$  in the definition of  $\mathcal{R}^\sharp f$ .

Any irrational number  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  can be written as an accelerated *continued fraction* of the form:

$$\alpha = a_0 + \frac{\varepsilon_0}{a_1 + \frac{\varepsilon_1}{a_2 + \frac{\varepsilon_2}{\ddots}}}, \text{ where } a_n \in \mathbb{Z}, \varepsilon_n = \pm 1 (n = 0, 1, 2, \dots), a_n \geq 2 (n \geq 1).$$

Let  $\mathbf{a}(x)$  be the closest integer to  $x \in \mathbb{R}$  (a convention:  $\mathbf{a}(m + \frac{1}{2}) = m$  for  $m \in \mathbb{Z}$ ) and  $T(x) = \mathbf{a}(\frac{1}{|x|}) = \frac{1}{|x|}$ . Then  $\alpha_n \in (-\frac{1}{2}, \frac{1}{2})$ ,  $a_n$  and  $\varepsilon_n$  are determined by

$$a_0 = \mathbf{a}(\alpha), \alpha_0 = \alpha - a_0, \varepsilon_0 = \text{sign } \alpha_0; a_{n+1} = \mathbf{a}(\frac{1}{|\alpha_n|}), \alpha_{n+1} = T(\alpha_n), \varepsilon_n = -\text{sign } \alpha_n. \quad (2.2)$$

**Successive renormalization.** Let  $f(z) = e^{2\pi i \alpha} z + \mathcal{O}(z^2)$  with  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  as above. The successive renormalization of  $f$  is defined as

$$f_0(z) = f(z) \text{ and } f_{n+1}(z) = \begin{cases} \mathcal{R}f_n(z) & (\alpha_n > 0) \\ \mathcal{R}f_n^*(z) & (\alpha_n < 0) \end{cases} \quad (n \geq 0), \quad (2.3)$$

where  $f_n(z) = e^{2\pi i \alpha_n} z + \mathcal{O}(z^2)$  with  $\alpha_n$  defined by (2.2). In general, it is not known that (2.3) is well defined or not. The second part in Theorem 1.2 gives an affirmative answer to this question.

**Definition** (Fiber renormalization). Write  $f$  as  $f(z) = e^{2\pi i \alpha} h(z)$ , where  $h(0) = 0$  and  $h'(0) = 1$ , we then identify  $f$  with the pair  $(\alpha, h)$ . The near-parabolic renormalization can be expressed as a skew product:

$$\mathcal{R} : (\alpha, h) \mapsto (T(\alpha), \mathcal{R}_\alpha h),$$

where  $\mathcal{R}_\alpha$  is the *renormalization in fiber direction* defined by

$$\mathcal{R}_\alpha h = \begin{cases} \text{Exp}^\sharp \circ E_{(e^{2\pi i \alpha} h)} \circ (\text{Exp}^\sharp)^{-1} & \text{if } \alpha \in (0, \frac{1}{2}] \\ \text{Exp}^\sharp \circ E_{(e^{-2\pi i \alpha} h)^\star} \circ (\text{Exp}^\sharp)^{-1} & \text{if } \alpha \in (-\frac{1}{2}, 0). \end{cases}$$

By the continuity of the horn maps, we have  $\mathcal{R}_\alpha h \rightarrow \mathcal{R}_0 h$  if  $|\arg \alpha| < \frac{\pi}{4}$  and  $\alpha \rightarrow 0$ . (Or  $\mathcal{R}_\alpha h \rightarrow \mathcal{R}_0 h^\star$  if  $|\arg(-\alpha)| < \frac{\pi}{4}$  and  $\alpha \rightarrow 0$ .) Therefore, this requires us to study the limiting case: the parabolic renormalization  $\mathcal{R}_0$ . This indicates the importance of the Main Theorem.

### 3. PROOF OF THE MAIN THEOREM

The main goal in this paper is to prove (b) of the Main Theorem, i.e. to find  $\psi$  such that  $\mathcal{R}_0 f = \Psi_0 \circ E_f \circ \Psi_0^{-1} = P \circ \psi^{-1}$ , where  $\Psi_0(z) = c \text{Exp}^\sharp(z)$  for some constant  $c \in \mathbb{C}^*$ . For the strategy about how to find  $\psi$  in Inou-Shishikura's case, see the first two paragraphs in [IS08, §5.A]. We will adapt their main strategy and work it in details in the following subsections.

**3.1. Outline of the proof.** Inou and Shishikura introduced a notion “subcover” in [IS08], which was used to analyze the covering properties of the maps in their class. The map  $P(z) = z(1 + \frac{2\sqrt{5}}{3}z + z^2)^2$  we introduced in the introduction has more complicated covering properties than  $z \mapsto z(1+z)^2$  defined in [IS08] since not all critical points of  $P$  are on the real line. Therefore, in order to analyze the covering properties, we first need to give some preparations. Recall that  $c_\pm = -\frac{\sqrt{5}}{3} \pm \frac{2}{3}i$  are two critical points of  $P$  satisfying  $P(c_+) = P(c_-) = 0$  and  $\mathbb{R}_- = (-\infty, 0)$  is the negative real line.

**Definition** ( $\ell_\pm$ ,  $\gamma_\pm$  and  $U_\pm$ ). Define two closed half-lines starting from  $c_\pm$  with slope  $\mp \frac{3+\sqrt{5}}{2}$ :

$$\ell_\pm : y \mp \frac{2}{3} = \mp \frac{3+\sqrt{5}}{2} \left( x + \frac{\sqrt{5}}{3} \right), \text{ where } x \leq -\frac{\sqrt{5}}{3}.$$

There exists a unique *unbounded* component  $\gamma_+$  of  $P^{-1}(\mathbb{R}_-)$  such that its closure  $\bar{\gamma}_+ = \gamma_+ \cup \{c_+\}$  is tangent to  $\ell_+$  at  $c_+$  and  $\ell_+ \cap \bar{\gamma}_+ = \{c_+\}$ . Moreover,  $\ell_+ \cup \gamma_+ \subset \mathbb{H}^+$ . We use  $U_+ \subset \mathbb{H}^+$  to denote the unbounded open domain bounding by  $\ell_+ \cup \gamma_+$  (see Figure 1 and Lemma 3.16).

Let  $\gamma_- = \{z | \bar{z} \in \gamma_+\}$  be the complex conjugacy of  $\gamma_+$ . Then  $\gamma_-$  is a component of  $P^{-1}(\mathbb{R}_-)$  such that its closure  $\bar{\gamma}_-$  is tangent to  $\ell_-$  at  $c_-$  and  $\ell_- \cap \bar{\gamma}_- = \{c_-\}$ . Let  $U_- \subset \mathbb{H}^-$  be the reflection of  $U_+$  about the real axis. Then  $U_-$  is the unbounded open domain bounding by  $\ell_- \cup \gamma_-$ . The half lines  $\ell_+$  and  $\ell_-$  lie very close to the curves  $\gamma_+$  and  $\gamma_-$  respectively in the spherical metric (see Figure 8 and §3.5). Denote  $\Upsilon := \mathbb{C} \setminus (\bar{U}_+ \cup \bar{U}_-)$ .

**Definition** (Classes  $\hat{\mathcal{F}}_0$ ,  $\tilde{\mathcal{F}}_0$  and  $\mathcal{F}_0$ ). We define three classes of maps:

$$\begin{aligned} \hat{\mathcal{F}}_0 &= \{ f = P \circ \varphi^{-1} \mid \varphi : \Upsilon \rightarrow \mathbb{C} \text{ is a normalized univalent mapping} \}; \\ \tilde{\mathcal{F}}_0 &= \{ f = P \circ \varphi^{-1} \mid \varphi : \mathbb{C} \setminus (\bar{\gamma}_+ \cup \bar{\gamma}_-) \rightarrow \mathbb{C} \text{ is a normalized univalent mapping} \}; \text{ and} \\ \mathcal{F}_0 &= \left\{ f : \text{Dom}(f) \rightarrow \mathbb{C} \left| \begin{array}{l} 0 \in \text{Dom}(f) \text{ open } \subset \mathbb{C}, f \text{ is holomorphic in } \text{Dom}(f), \\ f(0) = 0, f'(0) = 1, f : \text{Dom}(f) \setminus \{0\} \rightarrow \mathbb{C}^* \text{ is a branched} \\ \text{covering map with a unique critical value } cv_f, \text{ all critical} \\ \text{points are of local degree 3} \end{array} \right. \right\}. \end{aligned}$$

Here a univalent mapping is called *normalized* if  $\varphi(0) = 0$  and  $\varphi'(0) = 1$ . Obviously  $\tilde{\mathcal{F}}_0 \subset \hat{\mathcal{F}}_0$ ,  $\tilde{\mathcal{F}}_0 \subset \mathcal{F}_0$  and the cubic polynomial  $z \mapsto z + \sqrt{3}z^2 + z^3$  belongs to  $\mathcal{F}_0$ .

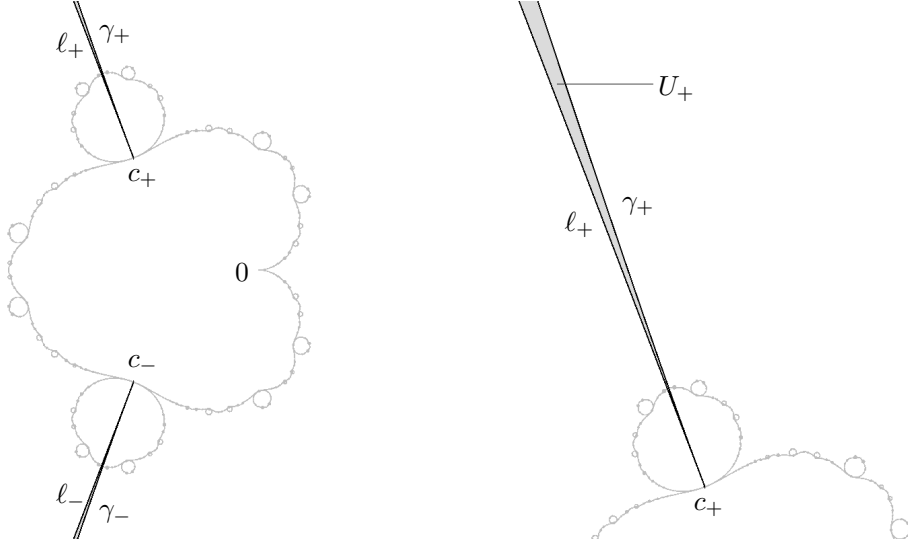


Figure 1: Left: The black curves ejecting from  $c_{\pm}$  are smooth curves  $\gamma_{\pm}$  and half lines  $\ell_{\pm}$ , where  $\gamma_{+}$  and  $\gamma_{-}$  are two components of  $P^{-1}(\mathbb{R}_{-})$  and  $\ell_{\pm}$  are tangent to  $\bar{\gamma}_{\pm}$  at  $c_{\pm}$  respectively. The light gray part is the Julia set of  $P$ ; Right: The zoom of the left picture near  $\ell_{+} \cup \gamma_{+}$  and the region  $U_{+}$  is marked.

We will describe the covering properties of the maps in  $\tilde{\mathcal{F}}_0$  and  $\mathcal{F}_0$  and establish a relation between these two classes in §3.3<sup>3</sup>. The idea is that the maps can be regarded as a (partial) branched covering over the ranges and they have similar covering structures when the maps restrict on certain “sheets”.

In [IS08], Inou and Shishikura established a connection between  $z \mapsto z(1+z)^2$  and a map with a parabolic fixed point at the infinity. In particular, they constructed a Riemann mapping from the outside of the closed unit disk onto  $\mathbb{C} \setminus (-\infty, -1]$ . However, we need to construct a Riemann mapping from  $\mathbb{C} \setminus \overline{\mathbb{D}}$  onto a double slitted complex plane. We cannot obtain the formula of a Riemann mapping defined from  $\mathbb{C} \setminus \overline{\mathbb{D}}$  onto  $\mathbb{C} \setminus (\bar{\gamma}_{+} \cup \bar{\gamma}_{-})$  since  $\gamma_{+}$  and  $\gamma_{-}$  are not “nice” boundaries. Instead, we use the complex plane deleted by two half lines  $\ell_{+} \cup \ell_{-}$  to replace the range of the Riemann mapping.

**Proposition 3.1.** *Define*

$$\psi_{1,2}(z) = \frac{\mu}{z^{\sigma}} \left( \frac{z^2}{2-\sigma} + \frac{1}{\sigma} \right) + \nu,$$

where

$$\sigma = \frac{2}{\pi} \arctan \frac{3+\sqrt{5}}{2}, \quad \mu = \frac{\sigma(\sigma-2)}{3} \sqrt{\frac{2}{1-\cos(\sigma\pi)}} e^{\frac{\sigma\pi i}{2}} \quad \text{and} \quad \nu = 1 - \frac{2\sqrt{5}}{3}.$$

Then  $\psi_{1,2}$  is a conformal mapping from the upper half plane  $\mathbb{H}^{+}$  onto  $\mathbb{C} \setminus (\ell_{+} \cup \ell_{-})$ . Moreover,  $\psi_{1,2}$  sends the boundary points  $-1, 0, 1, \infty$  to  $c_{+}, \infty, c_{-}, \infty$  respectively and  $\psi_{1,2}$  maps the upper half imaginary axis  $\{yi : y > 0\}$  onto the real line.

<sup>3</sup>There is a difference between our case and Inou-Shishikura’s. In their setting, if  $f \in \mathcal{F}_0$  (with local degree 2 in their definition) is not a quadratic polynomial, then  $f$  can be written as  $z(1+z)^2 \circ \varphi^{-1}$ , where  $\varphi : \mathbb{C} \setminus (-\infty, -1] \rightarrow \text{Dom}(f)$  is a normalized univalent map. However, in our case, because of the complexity of the covering properties of the maps in  $\mathcal{F}_0$ , we cannot guarantee that  $f \in \mathcal{F}_0$  can be written as  $P \circ \varphi^{-1}$  with  $\varphi : \mathbb{C} \setminus (\bar{\gamma}_{+} \cup \bar{\gamma}_{-}) \rightarrow \text{Dom}(f)$  a normalized univalent map if  $f$  is not a cubic polynomial. See §3.3 for more details.



The proof of Proposition 3.1 will be given in §3.4. The tool is Schwarz-Christoffel formula. According to Carathéodory, the Riemann mapping  $\psi_{1,2} : \mathbb{H}^+ \rightarrow \mathbb{C} \setminus (\ell_+ \cup \ell_-)$  can be extended continuously to a surjection  $\psi_{1,2} : \mathbb{H}^+ \cup \mathbb{R} \rightarrow \widehat{\mathbb{C}}$ .

**Definition** (Some intermediate mappings and the mapping  $Q$ ). Let  $\kappa i = \psi_{1,2}^{-1}(0)$  be the preimage of 0 under  $\psi_{1,2}$  which lies on the upper half imaginary axis, i.e.  $\kappa > 0$ . Define

$$\psi_{1,1}(z) = \kappa i \frac{z-1}{z+1}, \quad \psi_1(z) = \psi_{1,2} \circ \psi_{1,1}(z), \quad \psi_0(z) = -\frac{\tau}{z} \quad \text{and} \quad Q(z) = \psi_0^{-1} \circ P \circ \psi_1(z),$$

where  $\tau$  is the unique number such that  $Q$  has a 1-parabolic fixed point at  $\infty$ . The conformal mapping  $\psi_{1,1}$  maps the outsider of the closed unit disk onto the upper half plane  $\mathbb{H}^+$  and maps its closure  $\widehat{\mathbb{C}} \setminus \mathbb{D}$  isomorphically onto  $\mathbb{H}^+ \cup \mathbb{R}$ . This means that the map  $\psi_1$  and hence  $Q$  are defined only in  $\widehat{\mathbb{C}} \setminus \mathbb{D}$ . This is a difference between our setting and Inou-Shishikura's<sup>4</sup>. The full expression of  $Q$  can be obtained of course. But it is fairly long.

The floating point values of  $\kappa$  ( $\doteq 2.2142\dots$ ) and  $\tau$  ( $\doteq 2.1647\dots$ ) will be calculated in Lemmas 3.18 and 3.19 respectively.

**Definition** ( $V' = U_\eta^P$  and  $U_\eta^Q$ ). Let  $\eta > 0$  and  $cv_P = -\frac{64}{225\sqrt{5}}$  (which is a critical value of  $P$ ) and define

$$V' = U_\eta^P = P^{-1}(\mathbb{D}(0, |cv_P|e^{2\pi\eta})) \setminus \left( (\overline{U}_+ \cup \overline{U}_-) \cup \left( \text{the components of } P^{-1}(\overline{\mathbb{D}}(0, |cv_P|e^{-2\pi\eta})) \text{ containing } c_+ \text{ and } c_- \right) \right).$$

See Figure 2. Let  $U_\eta^Q = \psi_1^{-1}(U_\eta^P) \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ .

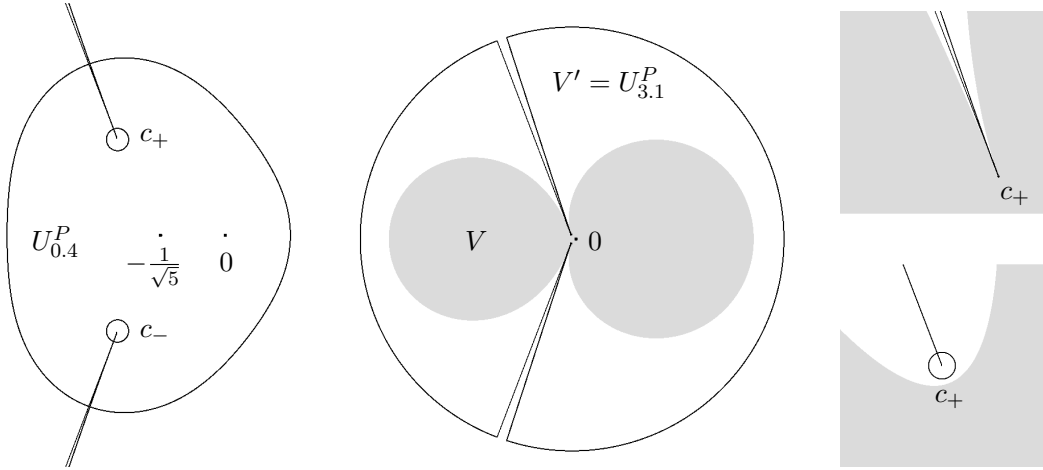


Figure 2: Left:  $U_\eta^P$  for  $\eta = 0.4$  (this  $\eta$  was chosen such that the deleted regions around  $c_+$  and  $c_-$  are visible); Middle:  $U_\eta^P$  for  $\eta = 3.1$  and  $V$ . The outer boundary of  $U_{3.1}^P$  looks like a round circle with radius about 32.6; Right: Successive zoom of  $U_{3.1}^P$  near  $c_+$  and the widths of the squares are 2 and 0.00024 respectively.

<sup>4</sup>Actually,  $Q$  can be defined in a more larger domain  $\widehat{\mathbb{C}} \setminus \{e^{i\theta} : \theta \in [0, \pi]\}$ . The reason is that  $\psi_{1,2}$  is holomorphic in  $\widehat{\mathbb{C}} \setminus [-\infty, 0]$  and  $\psi_{1,1}^{-1}([-\infty, 0]) = \{e^{i\theta} : \theta \in [0, \pi]\}$  (see the right picture in Figure 8). Sometimes we will use this fact to estimate the modulus of  $Q$  on the circles which are intersecting with the unit circle.

**Definition** (Ellipse  $E$  and  $V$ ). Let  $x_E = -0.053$ ,  $a_E = 1.057$ ,  $b_E = 1.029$  and define

$$E = \left\{ x + yi \in \mathbb{C} : \left( \frac{x - x_E}{a_E} \right)^2 + \left( \frac{y}{b_E} \right)^2 \leq 1 \right\}.$$

In Lemma 3.21 (b) we will prove  $\overline{\mathbb{D}} \subset \text{int } E$ . Then we can define  $V := \psi_1(\widehat{\mathbb{C}} \setminus E)$ .

**Proposition 3.2** (Relation between  $V$  and  $V'$ ). *Let  $\eta = 3.1$ . Then we have*

$$\widehat{\mathbb{C}} \setminus \text{int } E \subset U_\eta^Q \subset \psi_1^{-1}(\Upsilon) \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

Therefore, one has

$$\overline{V} \subset V' = U_\eta^P \subset \Upsilon = \widehat{\mathbb{C}} \setminus (\overline{U}_+ \cup \overline{U}_-).$$

The proof of this proposition will be given in §3.5 and the key ingredient is to prove  $\psi_1^{-1}(\overline{U}_+ \cup \overline{U}_-)$  is contained in the interior of the ellipse  $E$ . The constant  $\eta = 3.1$  and the ellipse  $E$  will be used throughout this paper. The reason we choose  $\eta = 3.1$  can be first observed by (3.60\*).

**Definition** (Classes  $\mathcal{F}_2^P$  and  $\mathcal{F}_1^Q$ ). From now on, we use  $\mathcal{F}_1^P$  to denote  $\mathcal{F}_1$ . We define two more classes of maps:

$$\begin{aligned} \mathcal{F}_2^P &= \{ f = P \circ \varphi^{-1} \mid \varphi : V' \rightarrow \mathbb{C} \text{ is a normalized univalent mapping} \}; \text{ and} \\ \mathcal{F}_1^Q &= \left\{ F = Q \circ \varphi^{-1} \left| \begin{array}{l} \varphi : \widehat{\mathbb{C}} \setminus E \rightarrow \widehat{\mathbb{C}} \setminus \{0\} \text{ is a normalized univalent mapping} \\ \text{and has a quasiconformal extension to } \widehat{\mathbb{C}} \end{array} \right. \right\}. \end{aligned}$$

Here a univalent mapping is called *normalized* if  $\varphi(\infty) = \infty$  and  $\lim_{z \rightarrow \infty} \frac{\varphi(z)}{z} = 1$  when  $\infty$  is in the domain.

**Proposition 3.3** (Relation between  $\mathcal{F}_1^P$ ,  $\mathcal{F}_2^P$ ,  $\mathcal{F}_1^Q$ ,  $\widehat{\mathcal{F}}_0$  and  $\widetilde{\mathcal{F}}_0$ ). *The following relation holds:*

$$\widetilde{\mathcal{F}}_0 \subset \widehat{\mathcal{F}}_0 \subset \mathcal{F}_2^P \subset \mathcal{F}_1^P \cong \mathcal{F}_1^Q.$$

It is formulated more precisely as follows:

- (a) *There is a natural injection  $\widetilde{\mathcal{F}}_0 \hookrightarrow \widehat{\mathcal{F}}_0 \hookrightarrow \mathcal{F}_2^P \hookrightarrow \mathcal{F}_1^P$ , defined by the restriction of  $\varphi$  to  $\Upsilon$ ,  $V'$  and  $V$  respectively for  $f = P \circ \varphi^{-1} \in \widetilde{\mathcal{F}}_0$ .*
- (b) *There exists a one to one correspondence between  $\mathcal{F}_1^P$  and  $\mathcal{F}_1^Q$ , defined by*

$$\begin{aligned} \mathcal{F}_1^P \ni f = P \circ \varphi^{-1} &\mapsto F = \psi_0 \circ f \circ \psi_0^{-1} = \psi_0^{-1} \circ P \circ \psi_1 \circ \psi_1^{-1} \circ \varphi^{-1} \circ \psi_0 = Q \circ \hat{\varphi}^{-1} \in \mathcal{F}_1^Q, \\ \text{with the correspondence } \varphi &\mapsto \hat{\varphi} = \psi_0^{-1} \circ \varphi \circ \psi_1. \text{ In this case, if } \hat{\varphi}(\zeta) = \zeta + c_0 + \mathcal{O}(\tfrac{1}{\zeta}) \\ \text{near } \infty, \text{ then } f''(0) &= 2(b_0 - c_0)/\tau, \text{ where } \tau \text{ and } b_0 \text{ are constants defined in Lemmas} \\ 3.19 \text{ and } 3.25 \text{ respectively.} \end{aligned}$$

The proof of this proposition will be given in §3.4. The value of  $b_0 (\doteq 7.3476\dots)$  is defined and calculated in Lemma 3.25. By Proposition 3.3 (b) and Lemma 3.28 (a), we have

$$\left| f''(0) - \frac{2(b_0 - 0.053)}{\tau} \right| = \frac{2|c_0 - 0.053|}{\tau} \leq \frac{2 \times 2.086}{\tau} (\doteq 1.9272\dots). \quad (3.1^*)$$

On the other hand, by a direct calculation, we have

$$\frac{2(b_0 - 0.053)}{\tau} (\doteq 6.7393\dots). \quad (3.2^*)$$

Therefore, we have  $|f''(0) - 6.74| \leq 1.93$  and the first part of Main Theorem (a) is proved.

In order to proof Main Theorem (b), it is sufficient to prove that if  $F = Q \circ \varphi^{-1} \in \mathcal{F}_1^Q$ , then the parabolic renormalization  $\mathcal{R}_0 F$  belongs to  $\mathcal{F}_2^P$ . In the following, we suppose that  $F = Q \circ \varphi^{-1} \in \mathcal{F}_1^Q$ . Then  $\varphi : \widehat{\mathbb{C}} \setminus E \rightarrow \widehat{\mathbb{C}} \setminus \{0\}$  is a normalized univalent mapping. As

in [IS08], we now define a Riemann surface  $X$  such that one can lift  $F^{-1} = \varphi \circ Q^{-1}$  to a single-valued branch on  $X$ .

**Definition** (Riemann surface  $X$ ). Let  $cv = cv_Q = \frac{225\sqrt{5}\tau}{64} (\doteq 17.0178\dots)$  (which is a critical value of  $Q$ , see Lemma 3.20),  $R = 100$  and  $\rho = 0.05$ . We define four “sheets” by

$$X_{1\pm} = \{z \in \mathbb{C} : \pm \operatorname{Im} z \geq 0, |z| > \rho \text{ and } \frac{\pi}{5} < \pm \arg(z - cv) \leq \pi\},$$

$$X_{2\pm} = \{z \in \mathbb{C} : z \notin \mathbb{R}_-, \pm \operatorname{Im} z \geq 0, \rho < |z| < R \text{ and } \frac{\pi}{5} < \pm \arg(z - cv) \leq \pi\}.$$

These “sheets” are considered as in the different copies of  $\mathbb{C}$  and we use  $\pi_{i\pm} : X_{i\pm} \rightarrow \mathbb{C}$  ( $i = 1, 2$ ) to denote the natural projection. The Riemann surface  $X$  is constructed as follows:  $X_{1+}$  and  $X_{1-}$  are glued along negative real axis,  $X_{1+}$  and  $X_{2-}$  are glued along positive real axis and  $X_{1-}$  and  $X_{2+}$  are also glued along positive real axis. The projection  $\pi_X : X \rightarrow \mathbb{C}$  is defined as  $\pi_X = \pi_{i\pm}$  on  $X_{i\pm}$  and the complex structure of  $X$  is given by the projection. See Figure 3.

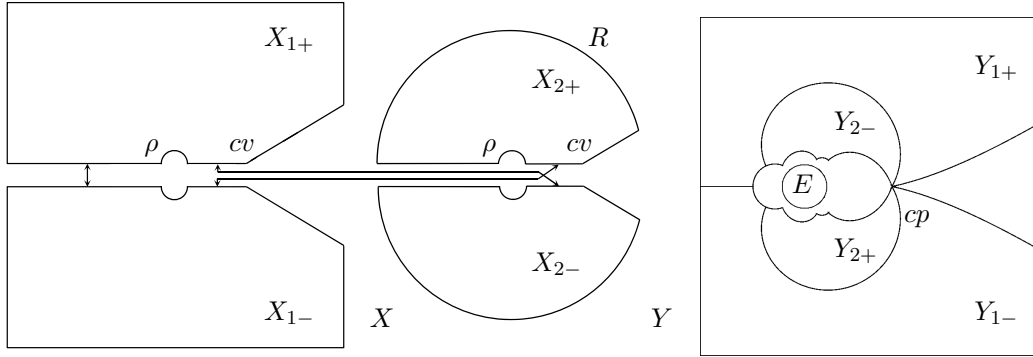


Figure 3: Riemann surfaces  $X$  (left) and domain  $Y$  (right).

**Proposition 3.4** (Lists of  $Q$  and  $\varphi$  to  $X$ ). *There exists an open subset  $Y \subset \mathbb{C} \setminus (E \cup \mathbb{R}_+)$  with the following properties:*

- (a) *There exists an isomorphism  $\tilde{Q} : Y \rightarrow X$  such that  $\pi_X \circ \tilde{Q} = Q$  on  $Y$  and  $\tilde{Q}^{-1}(z) = \pi_X(z) - b_0 + o(1)$  as  $z \in X$  and  $\pi_X(z) \rightarrow \infty$ ;*
- (b) *The normalized univalent map  $\varphi$  restricted to  $Y$  can be lifted to a univalent holomorphic map  $\tilde{\varphi} : Y \rightarrow X$  such that  $\pi_X \circ \tilde{\varphi} = \varphi$  on  $Y$ .*

This proposition will be proved in §3.8.

**Definition.** Define  $g = \tilde{\varphi} \circ \tilde{Q}^{-1} : X \rightarrow X$ . We have the following Figure 4.

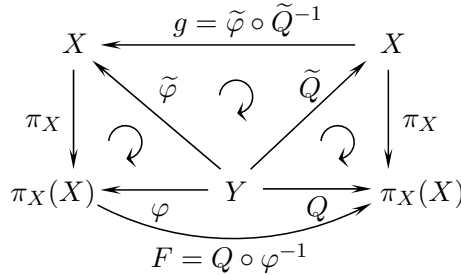


Figure 4: The commutative diagram about the maps  $Q$ ,  $\varphi$  and their lifts  $\tilde{Q}$ ,  $\tilde{\varphi}$ .

**Proposition 3.5** (Repelling Fatou coordinate on  $X$ ). *The map  $g$  satisfies  $F \circ \pi_X \circ g = \pi_X$ . There exists an injective holomorphic mapping  $\tilde{\Phi}_{rep} : X \rightarrow \mathbb{C}$  such that  $\tilde{\Phi}_{rep}(g(z)) = \tilde{\Phi}_{rep}(z) - 1$ . Moreover, in  $\{z : \operatorname{Re} z < -R\}$ ,  $\tilde{\Phi}_{rep} \circ \pi_X^{-1}$  is a repelling Fatou coordinate for  $F = Q \circ \varphi^{-1}$ .*

This proposition will be proved in §3.10.

**Definition.** For  $z_0 \in \mathbb{C}$  and  $\theta > 0$ , we denote  $\mathbb{V}(z_0, \theta) = \{z : z \neq z_0, |\arg(z - z_0)| < \theta\}$ ,  $\overline{\mathbb{V}}(z_0, \theta)$  = the closure of  $\mathbb{V}(z_0, \theta)$ . Define

$$W_1 = \mathbb{V}(cv, \frac{7\pi}{10}) \setminus \overline{\mathbb{V}}(F(cv), \frac{3\pi}{10}) = \{z : |\arg(z - cv)| < \frac{7\pi}{10} \text{ and } |\arg(z - F(cv)) - \pi| < \frac{7\pi}{10}\}.$$

We will prove in Lemma 3.33 that  $\operatorname{Re} F(cv) > 19 > cv$  and hence  $W_1$  is connected. Finally, let  $u_0 = 7.8/\cos(\frac{\pi}{5})$  and  $R_1 = 82$ .

**Proposition 3.6** (Attracting Fatou coordinate and the shape of  $D_1$ ). (a) *The map  $F$  maps the sector  $\mathbb{V}(u_0, \frac{7\pi}{10})$  into itself and  $\mathbb{V}(u_0, \frac{7\pi}{10})$  is contained in  $\operatorname{Basin}(\infty)$ . There exists an attracting Fatou coordinate  $\Phi_{attr} : \mathbb{V}(u_0, \frac{7\pi}{10}) \rightarrow \mathbb{C}$  such that  $\Phi_{attr}(F(z)) = \Phi_{attr}(z) + 1$  and  $\Phi_{attr}(cv) = 1$ . Moreover,  $\Phi_{attr}$  is injective in  $\mathbb{V}(u_0, \frac{7\pi}{10})$  and  $\Phi_{attr}(\mathbb{V}(u_0, \frac{7\pi}{10}))$  contains  $\{z : \operatorname{Re} z > 1\}$ .*

(b) *There are domains  $D_1, D_1^\sharp, D_1^\flat \subset W_1 (\subset \mathbb{V}(u_0, \frac{7\pi}{10}))$  such that*

$$\Phi_{attr}(D_1) = \{z : 1 < \operatorname{Re} z < 2, -\eta < \operatorname{Im} z < \eta\} \text{ and } D_1 \subset \mathbb{D}(cv, R_1);$$

$$\Phi_{attr}(D_1^\sharp) = \{z : 1 < \operatorname{Re} z < 2, \operatorname{Im} z > \eta\} \text{ and } D_1^\sharp \subset \{z : \frac{\pi}{5} < \arg(z - cv) < \frac{7\pi}{10}\};$$

$$\Phi_{attr}(D_1^\flat) = \{z : 1 < \operatorname{Re} z < 2, \operatorname{Im} z < -\eta\} \text{ and } D_1^\flat \subset \{z : -\frac{7\pi}{10} < \arg(z - cv) < -\frac{\pi}{5}\}.$$

This proposition will be proved in §3.11. The number  $\eta$  in this proposition can be replaced by any number between 3.1 and 5.4 while using the same  $R_1 = 82$  (see (3.60\*) and (3.61\*)). The above (a) implies that  $cv$  and  $cp_F = \varphi(cp_Q)$  are contained in  $\operatorname{Basin}(\infty)$ , which is the second half part of the Main Theorem (a). After adding a constant, we normalize  $\tilde{\Phi}_{rep}$  such that  $\tilde{\Phi}_{rep}(z) - \Phi_{attr}(\pi_X(z)) \rightarrow 0$  when  $z \in X$ ,  $\pi_X(z) \in D_1^\sharp$  and  $\operatorname{Im} \pi_X(z) \rightarrow +\infty$ .

**Proposition 3.7** (Domains around the critical point). *There exist disjoint Jordan domains  $D_0, D'_0, D''_0, D_{-1}, D'''_0, D''''_0$  and a domain  $D_0^\sharp$  such that*

- (a) *the closures  $\overline{D}_0, \overline{D}'_0, \overline{D}''_0, \overline{D}_{-1}, \overline{D}'''_0, \overline{D}''''_0$  and  $\overline{D}_0^\sharp$  are contained in  $\operatorname{Image}(\varphi) = \operatorname{Dom}(F)$ ;*
- (b)  *$F(D_0) = F(D'_0) = F(D''_0) = D_1, F(D_{-1}) = F(D'''_0) = F(D''''_0) = D_0$  and  $F(D_0^\sharp) = D_1^\sharp$ ;*
- (c)  *$F$  is injective on each of these domains;*
- (d)  *$cp_F = \varphi(cp_Q) \in \overline{D}_0 \cap \overline{D}'_0 \cap \overline{D}''_0 \cap \overline{D}_{-1} \cap \overline{D}'''_0 \cap \overline{D}''''_0, \overline{D}_0 \cap \overline{D}_1 \neq \emptyset, \overline{D}_0^\sharp \cap \overline{D}_1^\sharp \neq \emptyset$  and  $\overline{D}_{-1} \cap \overline{D}_0^\sharp \neq \emptyset$ ;*
- (e)  *$\overline{D}_0 \cup \overline{D}'_0 \cup \overline{D}''_0 \cup \overline{D}_{-1} \cup \overline{D}'''_0 \cup \overline{D}''''_0 \setminus \{cv\} \subset \pi_X(X_{2+}) \cup \pi_X(X_{2-}) = \mathbb{D}(0, R) \setminus (\mathbb{D}(0, \rho) \cup \mathbb{R}_- \cup \overline{\mathbb{V}}(cv, \frac{\pi}{5}))$  and  $\overline{D}_0^\sharp \subset \pi_X(X_{1+})$ .*

This proposition will be proved in §3.12. The main work there is to prove  $(\overline{D}_0 \cup \overline{D}'_0 \cup \overline{D}''_0 \cup \overline{D}_{-1} \cup \overline{D}'''_0 \cup \overline{D}''''_0) \cap \mathbb{R}_- = \emptyset$ . See Figure 5 for the shape of these domains in the case of  $\varphi = id$ .

**Proposition 3.8** (Relating  $E_F$  to  $P$ ). *The parabolic renormalization  $\mathcal{R}_0 F$  belongs to the class  $\mathcal{F}_2^P$  (possibly after a linear conjugacy). In fact, if we regard  $D_0, D'_0, D''_0, D'''_0, D''''_0, D_0^\sharp$  as subsets of  $X_{1+} \cup X_{1-} \subset X$  and let*

$$U = \text{the interior of } \bigcup_{n=0}^{\infty} g^n(\overline{D}_0 \cup \overline{D}'_0 \cup \overline{D}''_0 \cup \overline{D}'''_0 \cup \overline{D}''''_0 \cup \overline{D}_0^\sharp).$$

Then there exists a surjective holomorphic mapping  $\Psi_1 : U \rightarrow U_\eta^P \setminus \{0\} = V' \setminus \{0\}$  such that

- (a)  $P \circ \Psi_1 = \Psi_0 \circ \tilde{\Phi}_{attr}$  on  $U$ , where  $\Psi_0 : \mathbb{C} \rightarrow \mathbb{C}^*$ ,  $\Psi_0(z) = cv_P e^{2\pi iz} = cv_P \text{Exp}^\sharp(z)$ , and  $\tilde{\Phi}_{attr} : U \rightarrow \mathbb{C}$  is the natural extension of the attracting Fatou coordinate to  $U$ ;
- (b)  $\Psi_1(z) = \Psi_1(z')$  if and only if  $z' = g^n(z)$  or  $z = g^n(z')$  for some integer  $n \geq 0$ ;
- (c)  $\psi = \Psi_0 \circ \tilde{\Phi}_{rep} \circ \Psi_1^{-1} : V' \setminus \{0\} \rightarrow \mathbb{C}^*$  is well-defined and extends to a normalized univalent function on  $V'$ ;
- (d) on  $\psi(V' \setminus \{0\})$ , the following holds

$$P \circ \psi^{-1} = P \circ \Psi_1 \circ \tilde{\Phi}_{rep}^{-1} \circ \Psi_0^{-1} = \Psi_0 \circ \tilde{\Phi}_{attr} \circ \tilde{\Phi}_{rep}^{-1} \circ \Psi_0^{-1} = \Psi_0 \circ E_F \circ \Psi_0^{-1};$$

- (e) the holomorphic dependence holds as in Main Theorem (c).

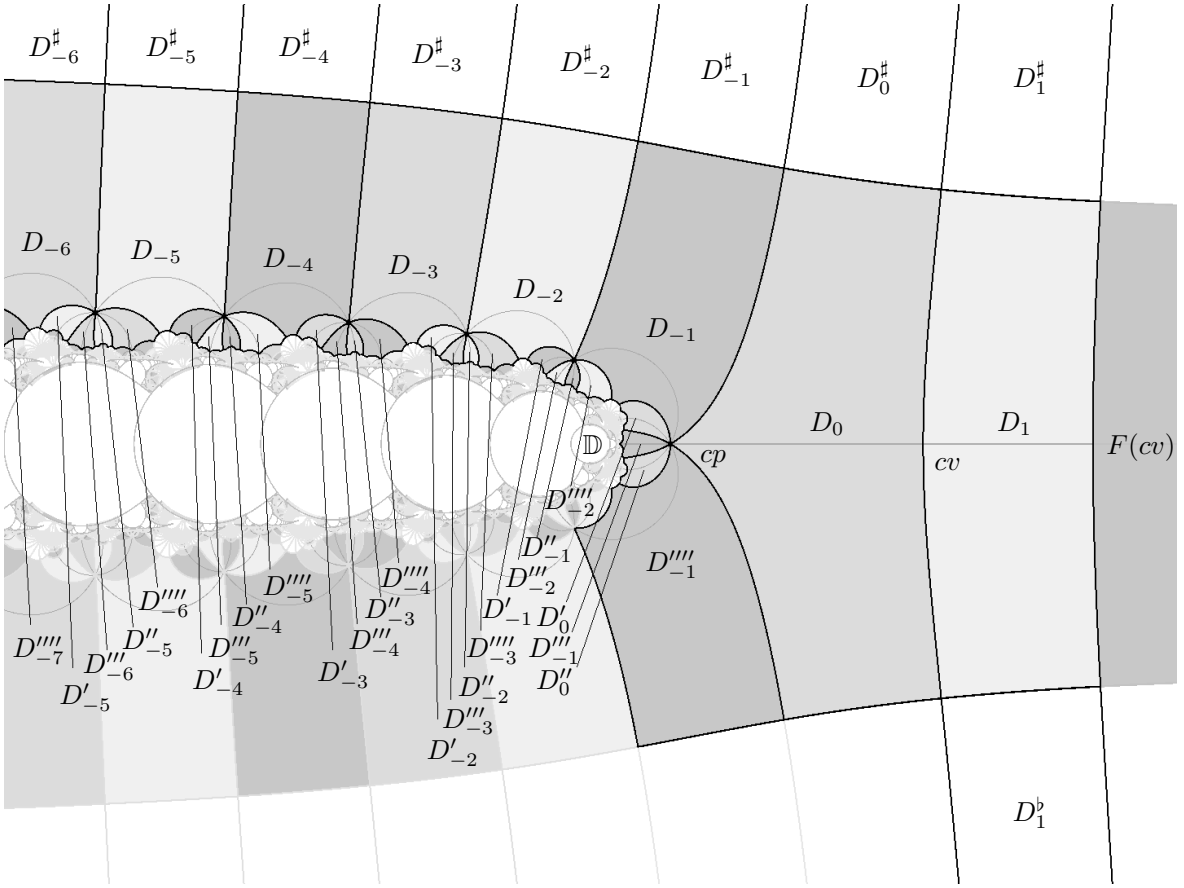


Figure 5: Domains  $D_1, D_0, D_1^\sharp$  etc. for  $F = Q$  (i.e.  $\varphi = id$ ). The inverse images were denoted by  $D_{-n} = g^n(D_0)$ ,  $D'_{-n} = g^n(D'_0)$ ,  $D''_{-n} = g^n(D''_0)$ ,  $D'''_{-n} = g^{n-1}(D'''_{-1})$ ,  $D''''_{-n} = g^{n-1}(D''''_{-1})$ ,  $D_{-n}^\sharp = g^n(D_0^\sharp)$ , and their projection by  $\pi_X$  were drawn. We deleted the preimages of the unit disk since  $Q$  has no definition in  $\mathbb{D}$ . In order to see the details,  $\eta = 1.5$  was chosen in this figure.

This proposition will be proved in §3.13. The map  $\Psi_1$  is defined by choosing an appropriate branch of  $P^{-1} \circ \Psi_0 \circ \tilde{\Phi}_{attr}$  on each domain  $D_{-n} = g^n(D_0)$  etc. After proving the consistency of  $\Psi_1$  (compare Figure 5 and Figure 6), we define

$$\mathcal{R}_0 F = P \circ \psi^{-1} \in \mathcal{F}_2^P \text{ for } F = Q \circ \varphi^{-1} \in \mathcal{F}_1^Q (\simeq \mathcal{F}_1^P).$$

Then (b) and (c) in the Main Theorem hold since  $\mathcal{F}_2^P \hookrightarrow \mathcal{F}_1^P$  by Proposition 3.3. This concludes the proof of the Main Theorem.

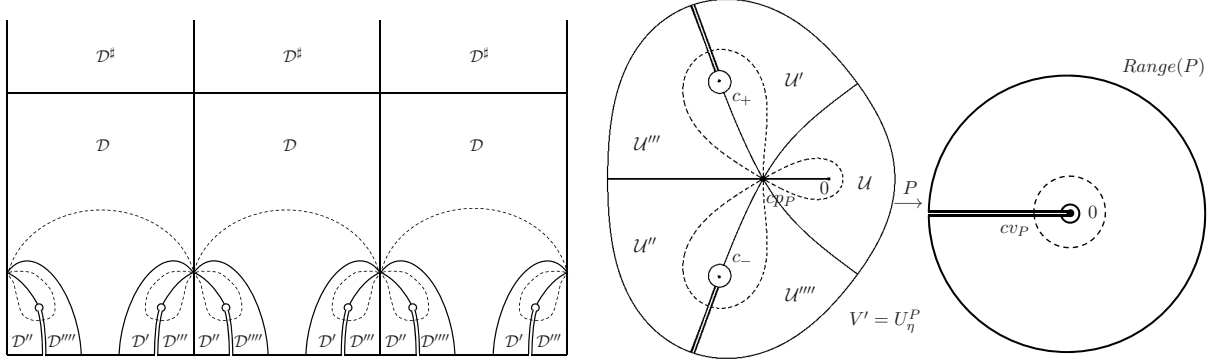


Figure 6: Domain  $U_\eta^P$  and its log lift (the inverse image of  $c_P \text{Exp}^\sharp$ ). Note that these pictures are just topologically correct but not conformally precise. We choose  $\eta = 0.4$  here such that the details are visible.

**3.2. Some useful lemmas.** We prepare some useful lemmas in this section. For the proofs of Lemma 3.9 to Theorem 3.12, see [IS08, §5.B].

**Lemma 3.9.** (a) If  $a, b \in \mathbb{C}$  and  $|a| > |b|$ , then  $|\arg(a + b) - \arg a| \leq \arcsin\left(\frac{|b|}{|a|}\right)$ . In particular, if  $|b| < 1$ , then  $|\arg(1 + b)| \leq \arcsin |b|$ .  
 (b) If  $0 \leq x \leq \frac{1}{2}$ , then  $\arcsin x \leq \frac{\pi}{3}x$ .

**Lemma 3.10.** Let  $e_1 = 1.043$ ,  $e_0 = -0.053 = x_E$ ,  $e_{-1} = 0.014$  and define

$$\zeta(w) = e_1 w + e_0 + \frac{e_{-1}}{w}.$$

Then  $\zeta$  is a conformal map from  $\mathbb{C} \setminus \overline{\mathbb{D}}$  onto  $\mathbb{C} \setminus E$ , and sends  $\{w : |w| = r\}$  onto  $\partial E_r$ , where  $E_r = \{x + yi \in \mathbb{C} : \left(\frac{x - e_0}{a_E(r)}\right)^2 + \left(\frac{y}{b_E(r)}\right)^2 \leq 1\}$  with  $a_E(r) = e_1 r + \frac{e_{-1}}{r}$  and  $b_E(r) = e_1 r - \frac{e_{-1}}{r}$ . For  $r = 1$ , we have  $a_E(1) = a_E$ ,  $b_E(1) = b_E$  and  $E_1 = E$ , which are defined in the last subsection.

**Lemma 3.11.** (a) If  $\text{Re}(ze^{-i\theta}) > t > 0$  with  $\theta \in \mathbb{R}$ , then

$$\frac{1}{z} \in \mathbb{D}\left(\frac{e^{-i\theta}}{2t}, \frac{1}{2t}\right);$$

(b) If  $H = \{z : \text{Re}(ze^{-i\theta}) > t\}$  and  $z_0 \in H$  with  $u = \text{Re}(z_0 e^{-i\theta}) - t$ , then

$$\mathbb{D}_H(z_0, s(r)) = \mathbb{D}\left(z_0 + \frac{2ur^2 e^{i\theta}}{1 - r^2}, \frac{2ur}{1 - r^2}\right),$$

where the right hand side is an Euclidean disk and  $s(r) = d_{\mathbb{D}}(0, r) = \log \frac{1+r}{1-r}$ .

**Theorem 3.12** (A general estimate on Fatou coordinate). Let  $\Omega$  be a disk or a half plane and  $f : \Omega \rightarrow \mathbb{C}$  a holomorphic function with  $f(z) \neq z$ . Suppose  $f$  has a univalent Fatou coordinate  $\Phi : \Omega \rightarrow \mathbb{C}$ , i.e.  $\Phi(f(z)) = \Phi(z) + 1$  when  $z, f(z) \in \Omega$ . If  $z \in \Omega$  and  $f(z) \in \Omega$ , then

$$\left| \log \Phi'(z) + \log(f(z) - z) - \frac{1}{2} \log f'(z) \right| \leq \log \cosh \frac{d_\Omega(z, f(z))}{2} = \frac{1}{2} \log \frac{1}{1 - r^2},$$

where  $r$  is a real number such that  $0 \leq r < 1$  and  $d_{\mathbb{D}}(0, r) = d_\Omega(z, f(z))$ .

The following lemma was claimed in the proof of [IS08, Lemma 5.31 (b)]. We include a detailed proof here by following the idea of Inou and Shishikura.

**Lemma 3.13.** *Let  $f : U \rightarrow \mathbb{C}$  be a holomorphic map with non-vanishing derivative which is defined on the domain  $U \subset \mathbb{C}$  and let  $[z_1, z_2] \subset U$  be a non-degenerate closed segment. If  $\theta$  and  $\theta'$  are two real numbers such that  $\theta < \arg f'(z) < \theta' \leq \theta + \pi$  on  $[z_1, z_2]$ , then*

$$\theta < \arg \frac{f(z_2) - f(z_1)}{z_2 - z_1} < \theta'.$$

*Proof.* Since  $z_1 \neq z_2$  and  $[z_1, z_2] \subset U$ , we have

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \frac{1}{z_2 - z_1} \int_0^1 \frac{d}{dt} f(z_1 + t(z_2 - z_1)) dt = \int_0^1 f'(z_1 + t(z_2 - z_1)) dt. \quad (3.3)$$

Define  $f_1(z) = e^{-i\theta - \frac{i\pi}{2}} f(z)$ . Then  $\arg f_1'(z) = -\theta - \frac{\pi}{2} + \arg f'(z) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . Hence  $\operatorname{Re} f_1'(z) > 0$ . By (3.3), we have

$$\operatorname{Re} \left( \frac{f_1(z_2) - f_1(z_1)}{z_2 - z_1} \right) = \operatorname{Re} \left( e^{-i\theta - \frac{i\pi}{2}} \cdot \frac{f(z_2) - f(z_1)}{z_2 - z_1} \right) > 0.$$

This is equivalent to

$$\theta = -\frac{\pi}{2} - \left( -\theta - \frac{\pi}{2} \right) < \arg \frac{f(z_2) - f(z_1)}{z_2 - z_1} < \frac{\pi}{2} - \left( -\theta - \frac{\pi}{2} \right) = \theta + \pi.$$

Similarly, after applying (3.3) to  $f_2(z) = e^{-i\theta' + \frac{i\pi}{2}} f(z)$ , one has

$$\theta' - \pi < \arg \frac{f(z_2) - f(z_1)}{z_2 - z_1} < \theta'.$$

The two inequalities above conclude this lemma.  $\square$

**Lemma 3.14.** *Let  $a = \alpha + \beta i \in \mathbb{C}^*$  and  $r > 0$  such that  $||a| - 1| < r < |a| + 1$ . Then  $\partial\mathbb{D} \cap \partial\mathbb{D}(a, r) = \{a + re^{i\theta_1}, a + re^{i\theta_2}\}$ , where  $\theta_1 := \theta_1(a, r)$  and  $\theta_2 := \theta_2(a, r)$  satisfy*

$$\cos \theta_{\{1,2\}} = \frac{\alpha A \mp |\beta| \sqrt{|a|^2 - A^2}}{|a|^2}, \sin \theta_{\{1,2\}} = \frac{\beta^2 A \pm \alpha |\beta| \sqrt{|a|^2 - A^2}}{\beta |a|^2} \text{ and } A = \frac{1 - |a|^2 - r^2}{2r}.$$

*In particular, if  $\beta = 0$ , i.e.  $a \in \mathbb{R} \setminus \{0\}$ , then*

$$\cos \theta_{\{1,2\}} = \frac{A}{a} \text{ and } \sin \theta_{\{1,2\}} = \pm \frac{\sqrt{a^2 - A^2}}{a}.$$

*Proof.* Consider the following equations:

$$\begin{cases} |z| = 1 \\ |z - a| = r \end{cases} \text{ i.e. } \begin{cases} z\bar{z} = 1 \\ z\bar{z} - \bar{a}z - a\bar{z} + |a|^2 = r^2, \end{cases}$$

we have  $2 \operatorname{Re}(\bar{a}z) = 1 + |a|^2 - r^2$ . Let  $z = a + re^{i\theta} \in \partial\mathbb{D}(a, r)$ . Then

$$2 \operatorname{Re}(\bar{a}z) = 2 \operatorname{Re}(|a|^2 + r \bar{a} e^{i\theta}) = 2|a|^2 + 2r(\alpha \cos \theta + \beta \sin \theta) = 1 + |a|^2 - r^2.$$

Therefore,

$$\alpha \cos \theta + \beta \sin \theta = \frac{1 - |a|^2 - r^2}{2r} = A.$$

Combine  $\cos^2 \theta + \sin^2 \theta = 1$ , one can obtain  $\theta_1$  and  $\theta_2$  as stated in the lemma. It is easy to check that  $|a|^2 - A^2 > 0$  if and only if  $||a| - 1| < r < |a| + 1$ . The proof is complete.  $\square$

**Computer Checked Inequalities, Maximum and Minimum.** Since the formula of  $Q$  is difficult to deal with by hand (see Lemma 3.25), we need the computer to help us to check the inequalities, to calculate the maximum and minimum on the closed intervals. In the following, the inequalities checked by the computer are marked by the star “\*” in the equation numbers. As in [IS08], the approximate values are indicated as  $x \doteq 0.1234\dots$ , which means that  $x \in (0.1234, 0.1235)$ .

**List of constants.** We give the list of the constants that will be used in this paper in Table 1. The column “Location” in the table indicates the place where the constants appear first or where the explicit definitions of the constants are given.

Notations	Values	Locations	Notations	Values	Locations
$cp = cp_Q$	4.0843...	Lemma 3.20	$\sigma$	0.7677...	Prop. 3.1
$cv = cv_Q$	17.0178...	Lemma 3.20	$\mu$	$-0.1204\dots$ $-0.3153\dots i$	Prop. 3.1
$x_E = e_0$	$-0.053$	§3.1	$\nu$	$-0.4907\dots$	Prop. 3.1
$a_E$	1.057	§3.1	$\kappa$	2.2142...	Lemma 3.18
$b_E$	1.029	§3.1	$\tau$	2.1647...	Lemma 3.19
$e_1$	1.043	Lemma 3.10	$b_0$	7.3476...	Lemma 3.25
$e_{-1}$	0.014	Lemma 3.10	$b_1$	21.4270...	Lemma 3.25
$\eta$	3.1	§3.1	$u_0$	9.6413...	§3.11
$R$	100	Lemma 3.24	$u_1$	7.8	§3.11
$\rho$	0.05	Lemma 3.24	$u_2$	5.4	§3.11
$r_1$	1.25	Lemma 3.24	$u_3$	13.7677...	§3.11
$R_1$	82	§3.1	$u_4$	11.5	§3.11
$c_{\pm}$	$-\frac{\sqrt{5}}{3} \pm \frac{2}{3}i$	§1	$c_{\pm}^Q$	$0.6611\dots$ $\pm 0.7502\dots i$	Lemma 3.20
$\varepsilon_1$	0.0036	Lemma 3.21	$r_4$	0.53	Lemma 3.35
$\varepsilon_2$	0.1	Lemma 3.21	$c_{00}$	0.053	Lemma 3.28
$\varepsilon_3$	0.0056	Lemma 3.21	$c_{01,max}$	2.086	Lemma 3.28
$\varepsilon_4$	0.9	Lemma 3.22	$a_4$	$-0.24 + 0.64i$	Lemma 3.22
$\varepsilon_5$	0.5	Lemma 3.22	$a_5$	$0.88 + 0.24i$	Lemma 3.22
$\varepsilon_6$	0.41	Lemma 3.24	$a_6$	$0.71 + 0.89i$	Lemma 3.24
$\varepsilon_7$	1	Lemma 3.24	$a_7$	$-1.45$	Lemma 3.24

Table 1: List of constants and their locations.

**3.3. Covering property of  $f \in \mathcal{F}_0$  and  $P$  as a partial covering.** Let  $f \in \mathcal{F}_0$  and suppose that the critical point  $cv = cv_f$  of  $f$  is contained in  $\mathbb{R}_-$ . We denote  $\Gamma_a = (cv, 0)$ ,  $\Gamma_b = (-\infty, cv]$ ,  $\Gamma_c = (0, +\infty) \subset \mathbb{R}$ . Define  $\mathbb{C}_{slit} = \mathbb{C} \setminus (\{0\} \cup \Gamma_b \cup \Gamma_c)$  and recall that  $\mathbb{H}^{\pm} = \{z : \pm \operatorname{Im} z > 0\}$  denote the upper and lower half planes.

The description of the covering properties of  $f \in \mathcal{F}_0$  in the following is inspired by [IS08, §5.C]. Since  $\mathbb{C}_{slit}$  is simply connected and contains no critical values, the preimage  $f^{-1}(\mathbb{C}_{slit})$  consists of several connected components  $\mathcal{U}_i$ , where  $i \in I$  and  $I$  is an index set which is equal to  $\mathbb{N}$  or  $\{1, 2, \dots, n\}$ . Each of these components is mapped isomorphically onto  $\mathbb{C}_{slit}$  by  $f$ . Denote  $\mathcal{U}_{i\pm} = f^{-1}(\mathbb{H}^{\pm}) \cap \mathcal{U}_i$ ,  $\gamma_{ai} = f^{-1}(\Gamma_a) \cap \mathcal{U}_i$ ,  $\gamma_{bi\pm} = f^{-1}(\Gamma_b) \cap \overline{\mathcal{U}_{i\pm}}$  and  $\gamma_{ci\pm} = f^{-1}(\Gamma_c) \cap \overline{\mathcal{U}_{i\pm}}$ , where the closures are taken in  $\operatorname{Dom}(f)$ . See left picture in Figure 7.



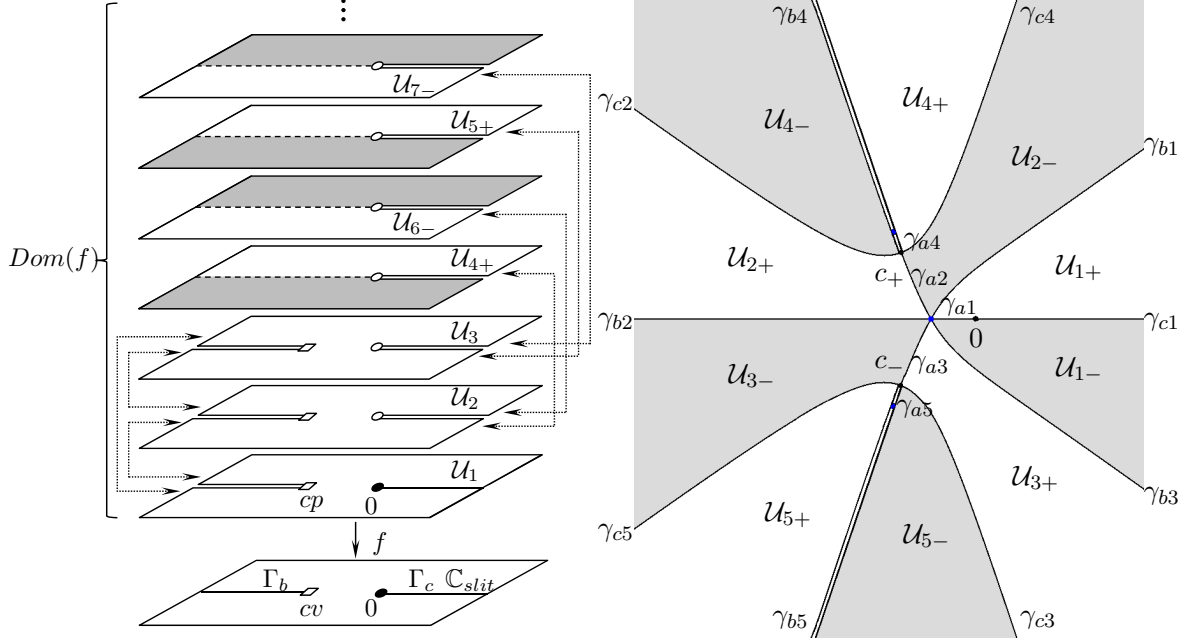


Figure 7: The domain  $Dom(f)$  as a Riemann surface spread over  $\mathbb{C}$  (left) and  $Dom(P)$  (right).

The domain  $Dom(f)$  can be described as the union of  $\bar{\mathcal{U}}_i$ 's which are glued along  $\gamma_{bi\pm}$  and  $\gamma_{ci\pm}$ . Each  $\gamma_{bi+}$  is glued with some  $\gamma_{bj-}$  and vice versa. This is also true for  $\gamma_{ci\pm}$ . If  $\gamma_{bi+}$  is glued with  $\gamma_{bj-}$  and  $\gamma_{bj+}$  is glued with  $\gamma_{bk-}$ , then  $\gamma_{bk+}$  is glued with  $\gamma_{bi-}$  since the critical point is of local degree 3. Because  $f$  is a homeomorphism near 0, there must exist a component, say  $\mathcal{U}_1$  such that  $0 \in \partial\mathcal{U}_1$  and  $\gamma_{c1+} = \gamma_{c1-}$ .

Now consider  $\gamma_{b1+}$  and  $\gamma_{b1-}$ . If they are glued together, then  $f$  has no critical value and this is a contradiction. Hence there must exist other two components  $\mathcal{U}_2$  and  $\mathcal{U}_3$  such that  $\gamma_{b1+} = \gamma_{b2-}$ ,  $\gamma_{b2+} = \gamma_{b3-}$  and  $\gamma_{b3+} = \gamma_{b1-}$ . Note that  $f^{-1}(cv) \cap \gamma_{b1+} \cap \gamma_{b1-}$  is a critical point with local degree 3. We call this point the *closest critical point* and denote it by  $cp = cp_f$ .

Denote  $\mathcal{U}_{123} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \gamma_{b1+} \cup \gamma_{b2+} \cup \gamma_{b3+}$ . Then  $f|_{\mathcal{U}_{123}} : \mathcal{U}_{123} \rightarrow \mathbb{C}_{slit} \cup \Gamma_b = \mathbb{C} \setminus (\{0\} \cup \Gamma_c)$  is a branched covering of degree 3 branched over  $cv_f$ .

**Example 1.** Let  $p(z) = z + \sqrt{3}z^2 + z^3$ ,  $Dom(p) = \mathbb{C} \setminus \{\frac{\sqrt{3}+i}{2}, \frac{\sqrt{3}-i}{2}\}$ . The critical point and critical value of  $p$  are  $cp = -\frac{1}{\sqrt{3}}$  and  $cv = -\frac{1}{3\sqrt{3}}$ . In this case,  $\mathcal{U}_1 = \mathbb{V}(-\frac{1}{\sqrt{3}}, \frac{\pi}{3}) \setminus [0, +\infty)$ ,  $\mathcal{U}_2 = e^{\frac{4\pi i}{3}}\mathcal{U}_1$  and  $\mathcal{U}_3 = e^{-\frac{4\pi i}{3}}\mathcal{U}_1$ .

**Example 2.** Let  $P(z) = z(1 + \frac{2\sqrt{5}}{3}z + z^2)^2$  and restricted to  $Dom(P) = \mathbb{C} \setminus \{c_+, c_-\}$ . The critical points are  $cp_P = -\frac{1}{\sqrt{5}}$  (with local degree 3) and  $c_{\pm} = -\frac{\sqrt{5}}{3} \pm \frac{2}{3}i$  and the critical values are  $cv_P = P(-\frac{1}{\sqrt{5}}) = -\frac{64}{225\sqrt{5}}$  and  $P(c_+) = P(c_-) = 0$ . It is easy to see  $\gamma_{a1} = (cp_P, 0)$ ,  $\gamma_{c1+} = \gamma_{c1-} = (0, +\infty)$  and  $\gamma_{b2+} = \gamma_{b3-} = (-\infty, cp_P)$ . The other preimages of  $\Gamma_a$ ,  $\Gamma_b$  and  $\Gamma_c$  must branch from  $cp_P$  and  $c_{\pm}$ . It can be checked that  $\gamma_{a2}$  (resp.  $\gamma_{a3}$ ) is a curve connecting  $cp_P$  with  $c_+$  (resp.  $c_-$ );  $\gamma_{a4}$  and  $\gamma_{a5}$  are two curves connecting  $c_+$  and  $c_-$  with two points  $P^{-1}(cv_P) \setminus \{cp_P\}$  respectively. The curves  $\gamma_{b1+} = \gamma_{b2-}$ ,  $\gamma_{c2-} = \gamma_{c4+}$ ,  $\gamma_{b4+} = \gamma_{b4-}$  and  $\gamma_{c4-} = \gamma_{c2+}$  divide the upper half plane  $\mathbb{H}^+$  into  $\mathcal{U}_{1+}$ ,  $\mathcal{U}_{2-}$ ,  $\mathcal{U}_{4+}$ ,  $\mathcal{U}_{4-}$  and  $\mathcal{U}_{2+}$ . The curves  $\gamma_{c3-} = \gamma_{c5+}$ ,  $\gamma_{b5+} = \gamma_{b5-}$ ,  $\gamma_{c5-} = \gamma_{c3+}$  and  $\gamma_{b3+} = \gamma_{b1-}$  divide the lower half plane  $\mathbb{H}^-$  into  $\mathcal{U}_{3-}$ ,  $\mathcal{U}_{5+}$ ,  $\mathcal{U}_{5-}$ ,  $\mathcal{U}_{3+}$  and  $\mathcal{U}_{1-}$ .

In the following, we denote  $\gamma_{bi} = \gamma_{bi+}$  and  $\gamma_{ci} = \gamma_{ci+}$  for simplicity. See right picture in Figure 7.

After a linear conjugacy, we assume further  $cv_f = -\frac{64}{225\sqrt{5}} = cv_P$ . Now we continue the description of  $Dom(f)$  for  $f \in \mathcal{F}_0$ . We already have three components  $\mathcal{U}_1, \mathcal{U}_2$  and  $\mathcal{U}_3$ . Now we consider  $\gamma_{c2+}, \gamma_{c2-}, \gamma_{c3+}$  and  $\gamma_{c3-}$ . If  $\gamma_{c2+}, \gamma_{c2-}$  are glued together and  $\gamma_{c3+}, \gamma_{c3-}$  are also, after adding two preimages of 0 to  $\mathcal{U}_2$  and  $\mathcal{U}_3$ , we will obtain a branched covering defined from  $\mathbb{C}$  to itself and this means that  $f$  is a cubic polynomial.

If  $f$  is not a cubic polynomial, there are following cases<sup>5</sup>:

(i) Any two of  $\gamma_{c2+}, \gamma_{c2-}, \gamma_{c3+}$  and  $\gamma_{c3-}$  are not glued together. There must exist four components  $\mathcal{U}_4, \mathcal{U}_5, \mathcal{U}_6$  and  $\mathcal{U}_7$  such that  $\gamma_{c2-} = \gamma_{c4+}, \gamma_{c2+} = \gamma_{c6-}, \gamma_{c3-} = \gamma_{c5+}$  and  $\gamma_{c3+} = \gamma_{c7-}$ . The further gluings for  $\gamma_{c4-}, \gamma_{c5-}, \gamma_{b4\pm}$  and  $\gamma_{b5\pm}$  etc. depend on particular  $f$ . See left picture in Figure 7.

(ii)  $\gamma_{c2+}$  and  $\gamma_{c2-}$  are glued together but  $\gamma_{c3+}$  and  $\gamma_{c3-}$  are not. There must exist two components  $\mathcal{U}_5$  and  $\mathcal{U}_7$  such that  $\gamma_{c3-} = \gamma_{c5+}$  and  $\gamma_{c3+} = \gamma_{c7-}$ . The further gluings for  $\gamma_{c5-}$  and  $\gamma_{b5\pm}$  etc. depend on particular  $f$ .

(iii)  $\gamma_{c3+}$  and  $\gamma_{c3-}$  are glued together but  $\gamma_{c2+}$  and  $\gamma_{c2-}$  are not. This case is similar to case (ii). There must exist two components  $\mathcal{U}_4$  and  $\mathcal{U}_6$  such that  $\gamma_{c2-} = \gamma_{c4+}$  and  $\gamma_{c2+} = \gamma_{c6-}$ . The further gluings for  $\gamma_{c4-}$  and  $\gamma_{b4\pm}$  etc. depend on particular  $f$ .

(iv)  $\gamma_{c2-}$  and  $\gamma_{c3+}$  are glued together but  $\gamma_{c2+}$  and  $\gamma_{c3-}$  are not. There must exist two components  $\mathcal{U}_5$  and  $\mathcal{U}_6$  such that  $\gamma_{c3-} = \gamma_{c5+}$  and  $\gamma_{c2+} = \gamma_{c6-}$ . The further gluings for  $\gamma_{c5-}$  and  $\gamma_{b5\pm}$  etc. depend on particular  $f$ .

(v)  $\gamma_{c2+}$  and  $\gamma_{c3-}$  are glued together but  $\gamma_{c2-}$  and  $\gamma_{c3+}$  are not. There must exist two components  $\mathcal{U}_4$  and  $\mathcal{U}_7$  such that  $\gamma_{c2-} = \gamma_{c4+}$  and  $\gamma_{c3+} = \gamma_{c7-}$ . The further gluings for  $\gamma_{c4-}$  and  $\gamma_{b4\pm}$  etc. depend on particular  $f$ .

(vi)  $\gamma_{c2-}$  and  $\gamma_{c3+}$  are glued together and  $\gamma_{c2+}$  and  $\gamma_{c3-}$  are also. Then  $Dom(f)$  is homeomorphic to an infinite cylinder after adding a preimage of 0.

**Proposition 3.15.** *The map  $f \in \mathcal{F}_0$  is contained in  $\tilde{\mathcal{F}}_0$  if and only if it has the covering properties stated in case (i).*

*Proof.* For case (i), although the components  $\mathcal{U}_4, \mathcal{U}_5, \mathcal{U}_6$  and  $\mathcal{U}_7$  for  $f$  may or may not be distinct, but  $f$  and  $P$  have the common structure up to the half components  $\mathcal{U}_{4+}, \mathcal{U}_{5+}, \mathcal{U}_{6-}$  and  $\mathcal{U}_{7-}$ . Denote the components and curves for  $P$  by  $\mathcal{U}_i^P$  and  $\gamma_{ai}^P$  etc. as in the right picture of Figure 7. Now we define  $\varphi : \mathbb{C} \setminus (\gamma_+ \cup \gamma_-) = \mathbb{C} \setminus (\gamma_{a4}^P \cup \gamma_{b4}^P \cup \gamma_{a5}^P \cup \gamma_{b5}^P) \rightarrow Dom(f)$  by  $\varphi(z) = (f|_{\mathcal{U}_{i\pm}})^{-1} \circ P(z)$  on  $\mathcal{U}_{i\pm}^P$  for  $i = 1, 2, 3, 4, 5$ , except on  $\mathcal{U}_{4-}^P$  and  $\mathcal{U}_{5-}^P$ , where  $(f|_{\mathcal{U}_{6-}})^{-1} \circ P$  and  $(f|_{\mathcal{U}_{7-}})^{-1} \circ P$  are used respectively. Since the gluing relation is the same (if the components  $\mathcal{U}_{4-}^P$  and  $\mathcal{U}_{5-}^P$  are denoted by  $\mathcal{U}_{6-}^P$  and  $\mathcal{U}_{7-}^P$  respectively), the definition of  $\varphi$  above can be extended continuously to the boundary curves  $\gamma_{b1}^P, \gamma_{b2}^P, \gamma_{b3}^P, \gamma_{c1}^P, \gamma_{c2}^P, \gamma_{c3}^P, \gamma_{c4}^P$  and  $\gamma_{c5}^P$ . The origin is mapped to itself and  $-\frac{1}{\sqrt{5}}$  is mapped to the closest critical point of  $f$  by  $\varphi$ . It is easy to see  $\varphi$  is a homeomorphism in  $\mathbb{C} \setminus (\gamma_+ \cup \gamma_-)$  and holomorphic except the union of finite number of analytic curves. By the removable singularity theorem, it follows that  $\varphi$  is conformal from  $\mathbb{C} \setminus (\gamma_+ \cup \gamma_-)$  onto its image. By the definition, we have  $f = P \circ \varphi^{-1}$ ,  $\varphi(0) = 0$  and  $\varphi'(0) = 1$ . Therefore, we have  $f \in \tilde{\mathcal{F}}_0$ .

For the rest five cases (ii)-(vi),  $f$  and  $P$  only have the common structure up to two of the half components  $\mathcal{U}_{4+}, \mathcal{U}_{5+}, \mathcal{U}_{6-}$  and  $\mathcal{U}_{7-}$ , but not all of them. Hence  $f$  cannot be written as  $f = P \circ \varphi^{-1}$  with  $\varphi : \mathbb{C} \setminus (\gamma_+ \cup \gamma_-) \rightarrow \mathbb{C}$  a normalized univalent map.  $\square$

By Proposition 3.15, we know that  $\tilde{\mathcal{F}}_0$  is a proper subset of  $\mathcal{F}_0 \setminus \{\text{cubic polynomials}\}$ . This is a difference between our case and Inou-Shishikura's. Although the maps in  $\mathcal{F}_0$  may have various kinds of covering properties, they just differ a univalent map after once parabolic renormalization. We will analyze this in §4.

<sup>5</sup>There is only one case in [IS08] if  $f$  is not a quadratic polynomial.

Recall that  $\ell_\pm$ ,  $\gamma_\pm$  and  $U_\pm$  are sets defined at the beginning of §3.1.

**Lemma 3.16.** *The closed curve  $\bar{\gamma}_+ = \gamma_+ \cup \{c_+\} := \gamma_{a4}^P \cup \gamma_{b4}^P \cup \{c_+\}$  is tangent to the half line  $\ell_+$  at  $c_+$  and  $\ell_+ \cap \bar{\gamma}_+ = \{c_+\}$ . Moreover,  $\ell_+ \setminus \{c_+\} \subset \mathcal{U}_{4-}^P$  and  $\bar{U}_+ \subset \mathbb{H}^+$ . The similar result holds for  $\gamma_-$ ,  $\ell_-$  and  $U_-$ .*

*Proof.* By definition,  $\gamma_{a4}^P \cup \gamma_{b4}^P$  is a component of  $P^{-1}(\mathbb{R}_-) = P^{-1}(\Gamma_a^P \cup \Gamma_b^P)$ . We first consider the set  $\{z : \text{Im } P(z) = 0\}$ . Suppose that  $z = x + yi$ , then a direct calculation shows that  $\text{Im } P(z) = \vartheta(x, y) = \frac{1}{9} y \vartheta_1(x, y) = 0$ , where

$$\vartheta_1(x, y) = 9y^4 - (90x^2 + 48\sqrt{5}x + 38)y^2 + 45x^4 + 48\sqrt{5}x^3 + 114x^2 + 24\sqrt{5}x + 9.$$

Note that the equation  $\vartheta_1(x, y) = 0$  has 4 solutions since it can be seen as a polynomial in terms of  $y$  with degree 4. Two of the solutions above the real axis are

$$y = h_\pm(x) = \frac{1}{3} \sqrt{45x^2 + 24\sqrt{5}x + 19 \pm 2|3x + \sqrt{5}| \sqrt{45x^2 + 18\sqrt{5}x + 14}}.$$

On the one hand, one can check that the figure  $\{(x, h_-(x)) : x \in \mathbb{R}\}$  of the function  $y = h_-(x)$  is equal to  $\gamma_{c2}^P \cup \gamma_{a2}^P \cup \gamma_{b1}^P \cup \{c_+\}$  since  $h_-(x)$  has two non-derivable points. On the other hand, note that  $h_-(x) \leq h_+(x)$  and the figure  $\{(x, h_+(x)) : x \in \mathbb{R}\}$  of the function  $y = h_+(x)$  is just the boundary of  $\mathcal{U}_{4+}$  since  $h_+(x)$  has only one non-derivable point at  $-\frac{\sqrt{5}}{3}$ . Therefore, we have  $\gamma_{a4}^P \cup \gamma_{b4}^P \cup \{c_+\} = \{(x, h_+(x)) : x \leq -\frac{\sqrt{5}}{3}\}$ . Then, by a straightforward calculation, the left derivative of  $y = h_+(x)$  at  $x = -\frac{\sqrt{5}}{3}$  is

$$(h_+)'_-(-\frac{\sqrt{5}}{3}) = -\frac{3+\sqrt{5}}{2},$$

which is the slope of the half line  $\ell_+$ . Therefore,  $\bar{\gamma}_+ = \gamma_{a4}^P \cup \gamma_{b4}^P \cup \{c_+\}$  is tangent to  $\ell_+$  at  $c_+$ .

We now prove that  $\ell_+ \setminus \{c_+\} \subset \mathcal{U}_{4-}^P$ . Then  $\ell_+ \cap \bar{\gamma}_+ = \{c_+\}$  and  $\bar{U}_+ \subset \mathbb{H}^+$  are direct corollaries. Note that it is sufficient to prove that  $P(\ell_+ \setminus \{c_+\}) \subset \mathbb{H}^-$ , or equivalently,  $\text{Im } P(\ell_+ \setminus \{c_+\}) < 0$ . For  $x < -\frac{\sqrt{5}}{3}$ , the imaginary part of  $P(\ell_+ \setminus \{c_+\})$  can be written as a function with variable  $x$  as  $\vartheta(x, \ell_+(x)) = \frac{1}{9} \ell_+(x) \vartheta_1(x, \ell_+(x))$ , where

$$\vartheta_1(x, \ell_+(x)) = -\frac{3}{2} \left(x + \frac{\sqrt{5}}{3}\right)^3 \left(3(13 + 9\sqrt{5})x + 9 + 17\sqrt{5}\right).$$

Note that

$$3(13 + 9\sqrt{5}) \cdot (-\frac{\sqrt{5}}{3}) + 9 + 17\sqrt{5} = 4(\sqrt{5} - 9) < 0.$$

Hence  $\vartheta_1(x, \ell_+(x))$  is negative on  $(-\infty, -\frac{\sqrt{5}}{3})$ . Note that  $\ell_+(x) > 0$  if  $x < -\frac{\sqrt{5}}{3}$ . Therefore,  $\text{Im } P(\ell_+ \setminus \{c_+\}) = \frac{1}{9} \ell_+(x) \vartheta_1(x, \ell_+(x)) < 0$  and  $\ell_+ \setminus \{c_+\} \subset \mathcal{U}_{4-}^P$ . The proof is complete.  $\square$

**3.4. Passing from  $P$  to  $Q$ .** In order to construct a Riemann mapping from the upper half plane onto a double slitted complex plane, we need the following Schwarz-Christoffel formula.

**Lemma 3.17** (Schwarz-Christoffel formula for a half-plane, [DT02, p. 10, Theorem 2.1]). *Let  $D$  be the interior of a polygon having vertices  $w_1, \dots, w_n$  and interior angles  $\alpha_1\pi, \dots, \alpha_n\pi$  in counterclockwise order. Let  $f$  be any conformal map from the upper half-plane  $\mathbb{H}^+$  onto  $D$  with  $f(\infty) = w_n$ . Then*

$$f(z) = \mu \int^z \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1} d\zeta + \nu$$

for some complex constants  $\mu$  and  $\nu$ , where  $w_k = f(z_k)$  for  $k = 1, \dots, n-1$ .

*Proof of Proposition 3.1.* We want to find a Riemann mapping  $\psi_{1,2}$  sending the upper half plane  $\mathbb{H}^+$  onto the double slitted plane  $\mathbb{C} \setminus (\ell_+ \cup \ell_-)$ . Since  $\mathbb{C} \setminus (\ell_+ \cup \ell_-)$  is symmetric about the real axis, we can specify the preimages of  $c_{\pm}$  and  $\infty$  in Table 2.

No. ( $k$ )	Vertices ( $w_k$ )	Angles ( $\alpha_k$ )	Preimages ( $z_k$ )
1	$c_+$	2	$-1$
2	$\infty$	$-\sigma$	0
3	$c_-$	2	$+1$
4	$\infty$	$-(2 - \sigma)$	$\infty$

Table 2: The information for constructing Riemann mapping from  $\mathbb{H}^+$  to  $\mathbb{C} \setminus (\ell_+ \cup \ell_-)$ , where  $\sigma = \frac{2}{\pi} \arctan \frac{3+\sqrt{5}}{2}$  is defined in Proposition 3.1.

Note that  $\sum_{k=1}^n \alpha_k = 2$  and  $n = 4$ . By Lemma 3.17, the map  $\psi_{1,2}$  can be written as

$$\psi_{1,2}(z) = \mu \int^z (\zeta + 1)(\zeta - 1)\zeta^{-\sigma-1} d\zeta + \nu = \frac{\mu}{z^\sigma} \left( \frac{z^2}{2 - \sigma} + \frac{1}{\sigma} \right) + \nu.$$

Solving the following equations

$$\begin{cases} \psi_{1,2}(+1) = c_- \\ \psi_{1,2}(-1) = c_+ \end{cases} \text{ i.e. } \begin{cases} \mu \left( \frac{1}{2-\sigma} + \frac{1}{\sigma} \right) + \nu = c_- \\ \mu \left( \frac{1}{2-\sigma} + \frac{1}{\sigma} \right) e^{-\sigma\pi i} + \nu = c_+, \end{cases}$$

we have

$$\mu = \frac{\sigma(\sigma - 2)}{3} \sqrt{\frac{2}{1 - \cos(\sigma\pi)}} e^{\frac{\sigma\pi i}{2}} \text{ and } \nu = 1 - \frac{2\sqrt{5}}{3}, \text{ where } \sigma = \frac{2}{\pi} \arctan \frac{3 + \sqrt{5}}{2}.$$

This ends the proof of Proposition 3.1.  $\square$

**Lemma 3.18.** *The point  $\kappa i = \psi_{1,2}^{-1}(0) (\doteq 2.2142 \dots i)$  is the solution of*

$$z^2 + \frac{\nu(2 - \sigma)}{\mu} z^\sigma + \frac{2 - \sigma}{\sigma} = 0. \quad (3.4)$$

*Proof.* By Proposition 3.1, we know that  $\psi_{1,2}^{-1}(0)$  is a number contained in the upper half imaginary axis. The equation (3.4) is obtained by simplifying  $\psi_{1,2}(z) = 0$  and the root

$$\kappa i = \psi_{1,2}^{-1}(0) (\doteq 2.2142 \dots i) \quad (3.5^*)$$

of (3.4) can be calculated easily by a computer.  $\square$

**Lemma 3.19.** *The constant  $\tau$  appeared in the definition of  $Q$  in §3.1 can be calculated by*

$$\tau = -2(\kappa^2 + 1) \cdot \frac{\mu}{(\kappa i)^\sigma} = \frac{2\nu\sigma(2 - \sigma)(\kappa^2 + 1)}{2 - \sigma(\kappa^2 + 1)} (\doteq 2.1647 \dots). \quad (3.6^*)$$

*Proof.* Consider the conjugacy  $\widehat{Q}(z) = \psi_0 \circ Q \circ \psi_0^{-1}$ , where  $\psi_0(z) = -\frac{\tau}{z}$ . Therefore

$$\widehat{Q}(z) = P \circ \psi_{1,2} \circ \psi_{1,1} \circ \psi_0^{-1}(z) = P \circ \psi_{1,2} \circ \widetilde{\psi}_{1,1}(z), \quad (3.7)$$

where

$$\widetilde{\psi}_{1,1}(z) = \psi_{1,1} \circ \psi_0^{-1}(z) = \kappa i \frac{\tau + z}{\tau - z}.$$

Then we have

$$\widehat{Q}'(z) = P'(\psi_{1,2} \circ \widetilde{\psi}_{1,1}(z)) \cdot \psi_{1,2}'(\widetilde{\psi}_{1,1}(z)) \cdot \widetilde{\psi}_{1,1}'(z). \quad (3.8)$$

Since  $Q$  has a 1-parabolic fixed point at  $\infty$ , then  $\widehat{Q}(z)$  has a 1-parabolic fixed point at the origin. Note that  $\widetilde{\psi}_{1,1}(0) = \kappa i$  and  $\psi_{1,2}(\kappa i) = 0$ . Therefore, we have

$$\widehat{Q}'(0) = P'(0) \cdot \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}'_{1,1}(0) = \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}'_{1,1}(0) = 1.$$

By a direct calculation, we have

$$\psi'_{1,2}(z) = \frac{\mu}{z^{1+\sigma}}(z^2 - 1) \text{ and } \widetilde{\psi}'_{1,1}(z) = \frac{2\tau\kappa i}{(\tau - z)^2}. \quad (3.9)$$

Therefore, we have

$$\widehat{Q}'(0) = \frac{\mu}{(\kappa i)^{1+\sigma}}(-\kappa^2 - 1) \cdot \frac{2\kappa i}{\tau} = -\frac{2(\kappa^2 + 1)}{\tau} \cdot \frac{\mu}{(\kappa i)^\sigma} = 1.$$

By Lemma 3.18, we have

$$\frac{\mu}{(\kappa i)^\sigma} = \frac{\nu\sigma(2 - \sigma)}{\sigma(\kappa^2 + 1) - 2}.$$

This means that

$$\tau = -2(\kappa^2 + 1) \cdot \frac{\mu}{(\kappa i)^\sigma} = \frac{2\nu\sigma(2 - \sigma)(\kappa^2 + 1)}{2 - \sigma(\kappa^2 + 1)}. \quad \square$$

**Lemma 3.20.** *The map  $Q$  has only one critical point*

$$cp := \psi_1^{-1}(-\frac{1}{\sqrt{5}}) (\doteq 4.0843 \dots) \quad (3.10^*)$$

in  $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  and four critical points<sup>6</sup>

$$c_\pm^Q := \psi_1^{-1}(c_\pm)(\doteq 0.6611 \dots \pm 0.7502 \dots i) \quad (3.11^*)$$

and  $\pm 1$  on the boundary of  $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ . Moreover, the critical value of  $Q$  is

$$cv := Q(cp) = \frac{225\sqrt{5}\tau}{64} (\doteq 17.0178 \dots). \quad (3.12^*)$$

*Proof.* The values of  $cp$  and  $c_\pm^Q$  can be calculated as in Lemma 3.18. By the definition in §3.1, we have  $Q(z) = \psi_0^{-1} \circ P \circ \psi_1(z)$ . Hence the critical value of  $Q$  is equal to the critical value of  $P$  after post-compositing  $\psi_0^{-1}$ . It was known that the critical value (besides the origin) of  $P$  is  $-\frac{64}{225\sqrt{5}}$  and  $\psi_0^{-1}(z) = -\frac{\tau}{z}$ . Therefore, the critical value of  $Q$  is  $cv := Q(cp) = \frac{225\sqrt{5}\tau}{64}$ .  $\square$

**Remark.** The mapping relation of some important points under  $Q = \psi_0^{-1} \circ P \circ \psi_1 = \psi_0^{-1} \circ P \circ \psi_{1,2} \circ \psi_{1,1}$  are

$$\begin{aligned} 1 &\xrightarrow{\psi_{1,1}} 0 \xrightarrow{\psi_{1,2}} \infty \xrightarrow{P} \infty \xrightarrow{\psi_0^{-1}} 0; \quad -1 \xrightarrow{\psi_{1,1}} \infty \xrightarrow{\psi_{1,2}} \infty \xrightarrow{P} \infty \xrightarrow{\psi_0^{-1}} 0; \\ c_\pm^Q &\xrightarrow{\psi_{1,1}} \mp 1 \xrightarrow{\psi_{1,2}} c_\pm \xrightarrow{P} 0 \xrightarrow{\psi_0^{-1}} \infty; \quad cp \xrightarrow{\psi_1} -\frac{1}{\sqrt{5}} \xrightarrow{P} -\frac{64}{225\sqrt{5}} \xrightarrow{\psi_0^{-1}} cv. \end{aligned} \quad (3.13)$$

**Definition.** Define  $\mathcal{U}_{i\pm}^Q = \psi_1^{-1}(\mathcal{U}_{i\pm}^P)$ ,  $\Gamma_a^Q = \psi_0^{-1}(\Gamma_a^P)$ ,  $\gamma_{ai}^Q = \psi_1^{-1}(\gamma_{ai}^P)$  etc. Then  $\Gamma_a^Q = (cv, +\infty)$ ,  $\Gamma_b^Q = (0, cv]$ ,  $\Gamma_c^Q = (-\infty, 0)$ ,  $\gamma_{a1}^Q = (cp, +\infty)$ ,  $\gamma_{b2}^Q = (1, cp)$ ,  $\gamma_{c1}^Q = (-\infty, -1)$ . Note that  $\psi_1^{-1}$  split the two half lines  $\ell^+$  and  $\ell^-$  into arcs on  $\partial\mathbb{D}$ . By Lemma 3.16, each  $\mathcal{U}_{i\pm}^Q$  is connected, except  $\mathcal{U}_{4-}^Q$  and  $\mathcal{U}_{5+}^Q$ . They both consist of two components. Let  $U_\pm^Q := \psi_1^{-1}(U_\pm) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$  and  $\Upsilon^Q := \psi_1^{-1}(\Upsilon) = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \overline{U}_+^Q \cup \overline{U}_-^Q)$ . Note that  $\partial U_+^Q = \gamma_{b4} \cup \gamma_{a4} \cup [-1, c_+^Q]_{\partial\mathbb{D}}$  and  $\partial U_-^Q = \gamma_{b5} \cup \gamma_{a5} \cup [-1, c_-^Q]_{\partial\mathbb{D}}$ . We use  $\mathcal{U}_{4-}^Q$  and  $\mathcal{U}_{5+}^Q$  to denote the

<sup>6</sup>Note that  $Q$  has no derivative at these four points since  $Q$  is only defined on  $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ . But we still call them critical points since  $Q$  is not injective in any neighborhood of them. Of course, if we consider the larger domain of  $Q$ , i.e. in  $\widehat{\mathbb{C}} \setminus \{e^{i\theta} : \theta \in [0, \pi]\}$ ,  $Q$  is holomorphic at  $c_-^Q$  but  $Q$  can never be holomorphic at  $\pm 1$  and  $c_+^Q$ .

components of  $\mathcal{U}_{4-}^Q$  and  $\mathcal{U}_{5+}^Q$  with boundary components  $[1, c_+^Q]_{\partial\mathbb{D}}$  and  $[1, c_-^Q]_{\partial\mathbb{D}}$  respectively. Then we have  $\mathcal{U}_{4-}^Q = \mathcal{U}_{4-}'^Q \cup U_+^Q$  and  $\mathcal{U}_{5+}^Q = \mathcal{U}_{5+}'^Q \cup U_-^Q$  (see Figure 8).

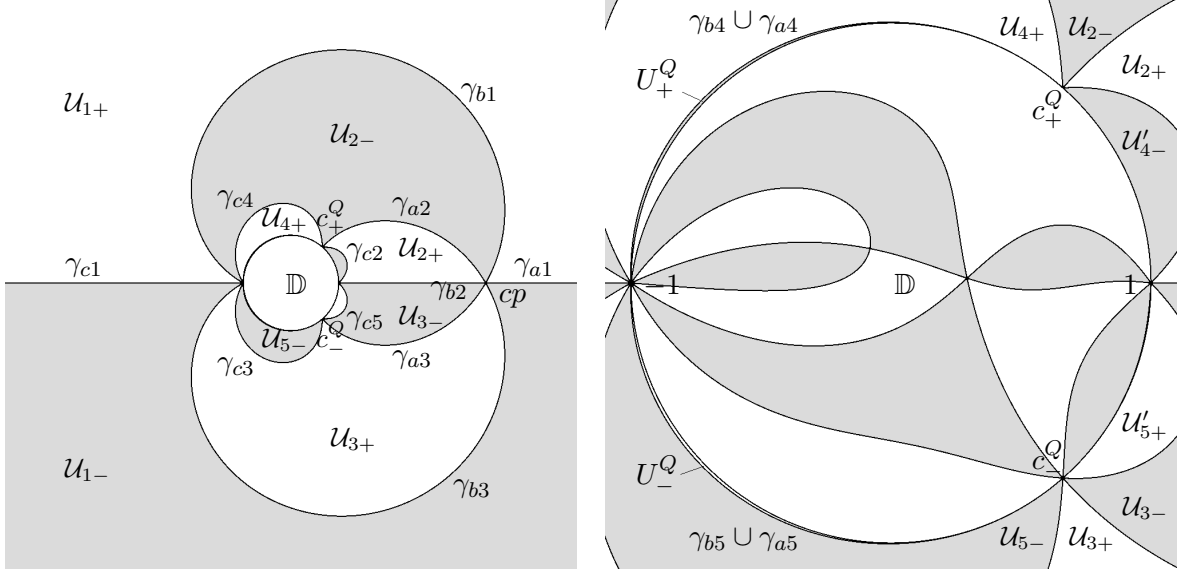


Figure 8: The domain of  $Q$  with partition by curves and its zoom near the unit circle. Note that the superscript “ $Q$ ” has been omitted in most of the notations. It can be seen in the right figure that the curves  $\psi_1^{-1}(\gamma_{\pm})$  lie very close to  $(-1, c_{\pm}^Q]_{\partial\mathbb{D}} \subset \psi_1^{-1}(\ell_{\pm})$ , i.e. the curves  $\gamma_{b4} \cup \gamma_{a4}$  and  $\gamma_{b5} \cup \gamma_{a5}$  are very close to the unit circle. Although  $Q$  has no definition in  $\mathbb{D}$ , but  $Q$  can be extended to a holomorphic map in  $\widehat{\mathbb{C}} \setminus \{e^{i\theta} : \theta \in [0, \pi]\}$  as stated in §3.1. The discontinuous of  $Q$  at upper half unit circle can be clearly seen by the mapping partition in this figure.

The map  $Q$  maps each  $\mathcal{U}_{i\pm}^Q$  (except  $\mathcal{U}_{4-}$  and  $\mathcal{U}_{5+}$ ) isomorphically onto  $\{z : \pm \operatorname{Im} z > 0\}$  and  $\gamma_{ai}^Q$  homeomorphically onto  $\Gamma_a^Q$  etc. Denote  $\mathcal{U}_{123}^Q = \mathcal{U}_1^Q \cup \mathcal{U}_2^Q \cup \mathcal{U}_3^Q \cup \gamma_{b1+}^Q \cup \gamma_{b2+}^Q \cup \gamma_{b3+}^Q = \psi_1^{-1}(\mathcal{U}_{123}^P)$ . Then  $Q|_{\mathcal{U}_{123}^Q} : \mathcal{U}_{123}^Q \rightarrow \mathbb{C}_{slit} \cup \Gamma_b^Q = \mathbb{C} \setminus (\{0\} \cup \Gamma_c^Q)$  is a branched covering of degree 3 branched over  $cv_Q$ .

*Proof of Proposition 3.3 assuming Proposition 3.2.* For (a) and first half part of (b), the reader can refer [IS08, §5.D] for a complete similar proof. We only prove the statement on  $f''(0)$ . Note that by Lemma 3.25, we can written  $Q(\zeta) = \zeta + b_0 + \mathcal{O}(\frac{1}{\zeta})$  as  $\zeta$  tends to  $\infty$ , where  $b_0 (\doteq 7.3476\dots)$  is a constant. Since  $\hat{\varphi}(\zeta) = \zeta + c_0 + \mathcal{O}(\frac{1}{\zeta})$  as  $\zeta$  tends to  $\infty$ , then  $F(z) = Q \circ \hat{\varphi}^{-1}(z) = z + (b_0 - c_0) + \mathcal{O}(\frac{1}{z})$  near  $\infty$ . Therefore,  $f(z) = \psi_0^{-1} \circ F \circ \psi_0(z) = -\tau / (-\tau/z + (b_0 - c_0) + \mathcal{O}(z)) = z + \frac{b_0 - c_0}{\tau} z^2 + \mathcal{O}(z^3)$ . Hence  $f''(0) = \frac{2(b_0 - c_0)}{\tau}$ .  $\square$

In the following subsections, if there is no confusion, we will drop the superscript “ $Q$ ” in the notation  $\mathcal{U}_i^Q, \gamma_{ai}^Q$  etc. and denote them by  $\mathcal{U}_i, \gamma_{ai}$  etc. for simplicity.

**3.5. Estimates on  $Q$ : Part I.** The numerical estimations will begin from this subsection. They are important ingredients in the proof of the Main Theorem (b). In the following two subsections, we will give some estimations on the map  $Q$ . Actually, among the estimations in the following (not only for  $Q$ , but also for  $\varphi$  and  $F$ ), most of them have corresponding figures for visualization. Therefore, we suggest that the reader can read the pictures first and then check the proofs.

Recall that  $U_\eta^Q = \psi_1^{-1}(U_\eta^P) = \psi_1^{-1}(V') \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  was defined in §3.1, the sets  $U_\pm^Q = \psi_1^{-1}(U_\pm)$  and  $\Upsilon^Q = \psi_1^{-1}(\Upsilon) = \widehat{\mathbb{C}} \setminus (\overline{\mathbb{D}} \cup \overline{U_+^Q} \cup \overline{U_-^Q})$  were defined in §3.4.

**Lemma 3.21.** *Let  $\eta = 3.1$ ,  $\varepsilon_1 = 0.0036$ ,  $\varepsilon_2 = 0.1$  and  $\varepsilon_3 = 0.0056$ .*

- (a)  $\Upsilon^Q \setminus U_\eta^Q$  is covered by the union of the disks  $\mathbb{D}(1, \varepsilon_1)$ ,  $\mathbb{D}(-1, \varepsilon_2)$ ,  $\mathbb{D}(c_+^Q, \varepsilon_3)$  and  $\mathbb{D}(c_-^Q, \varepsilon_3)$ .
- (b) The closed disks  $\overline{\mathbb{D}}$ ,  $\overline{\mathbb{D}}(1, \varepsilon_1)$ ,  $\overline{\mathbb{D}}(-1, \varepsilon_2)$ ,  $\overline{\mathbb{D}}(c_+^Q, \varepsilon_3)$  and  $\overline{\mathbb{D}}(c_-^Q, \varepsilon_3)$  are contained in the interior of the ellipse  $E$ .

See Figure 9.

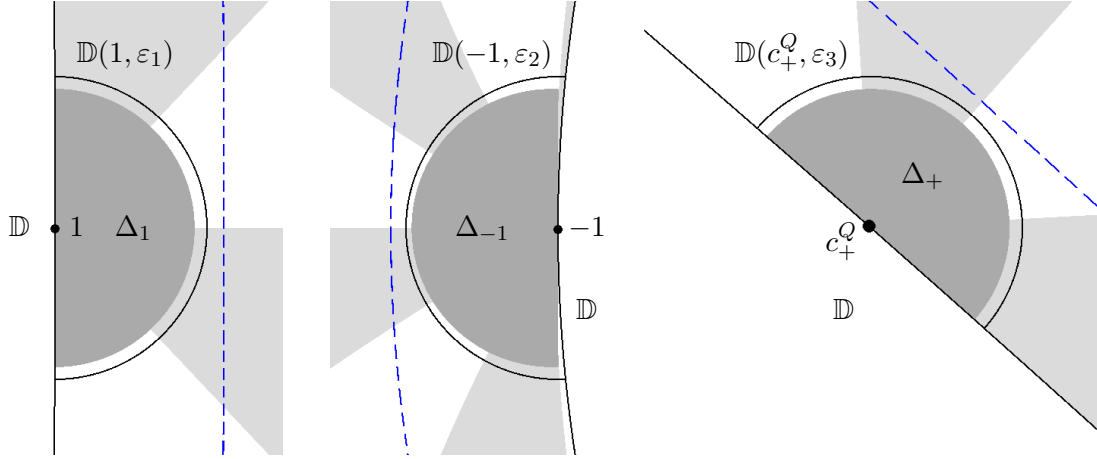


Figure 9: The zoom of the parts of the disks  $\mathbb{D}(1, \varepsilon_1)$ ,  $\mathbb{D}(-1, \varepsilon_2)$  and  $\mathbb{D}(c_+^Q, \varepsilon_3)$  near the unit circle (from left to right). It is clear that the regions in dark gray (which are the components of  $\Upsilon^Q \setminus U_\eta^Q$ ) are contained in the corresponding disks and these disks are contained in the ellipse  $E$  (the boundary of  $E$  is depicted by a dashed line). See Figure 11 for a global picture.

*Proof.* (a) By the definition of  $\Upsilon$  and  $U_\eta^P$ , we know that  $\Upsilon \setminus U_\eta^P$  consists three components and the boundaries of them contain  $\infty$ ,  $c_+$  and  $c_-$  respectively. Then according to the mapping properties of  $\psi_1 = \psi_{1,2} \circ \psi_{1,1}$  in (3.13), it follows that  $\Upsilon^Q \setminus U_\eta^Q = \psi_1^{-1}(\Upsilon \setminus U_\eta^P)$  consists of four connected components  $\Delta_1$ ,  $\Delta_{-1}$ ,  $\Delta_+$  and  $\Delta_-$  such that  $\Delta_1$  (resp.  $\Delta_{-1}$ ,  $\Delta_+$  and  $\Delta_-$ ) contains 1 (resp.  $-1$ ,  $c_+^Q$  and  $c_-^Q$ ) in its boundary and  $|Q(\zeta)| \leq cv e^{-2\pi\eta}$  in  $\Delta_1$  and  $\Delta_{-1}$  (resp.  $|Q(\zeta)| \geq cv e^{2\pi\eta}$  in  $\Delta_+$  and  $\Delta_-$ ). In the following, if we show that  $|Q(\zeta)| > cv e^{-2\pi\eta}$  on  $\partial\mathbb{D}(1, \varepsilon_1) \cup \partial\mathbb{D}(-1, \varepsilon_2) \setminus \mathbb{D}$  and  $|Q(\zeta)| < cv e^{2\pi\eta}$  on  $\partial\mathbb{D}(c_+^Q, \varepsilon_3) \cup \partial\mathbb{D}(c_-^Q, \varepsilon_3) \setminus \mathbb{D}$ , then  $\Delta_1 \subset \mathbb{D}(1, \varepsilon_1)$ ,  $\Delta_{-1} \subset \mathbb{D}(-1, \varepsilon_2)$  and  $\Delta_\pm \subset \mathbb{D}(c_\pm^Q, \varepsilon_3)$  follow immediately.

We first consider the minimum of  $|Q(\zeta)|$  on the circle  $\partial\mathbb{D}(1, \varepsilon_1)$  and have the following numerical estimate<sup>7</sup>

$$\begin{aligned} \min_{\zeta \in \partial\mathbb{D}(1, \varepsilon_1) \setminus \mathbb{D}} |Q(\zeta)| &\geq \min_{\theta \in [0, 2\pi]} |Q(1 + \varepsilon_1 e^{i\theta})| (\doteq 7.9800 \dots \times 10^{-8}) \\ &> cv e^{-2\pi\eta} (\doteq 5.9124 \dots \times 10^{-8}). \end{aligned} \quad (3.14^*)$$

<sup>7</sup>As we pointed out in §3.1,  $Q$  is holomorphic in  $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$  and is continuous on  $\widehat{\mathbb{C}} \setminus \mathbb{D}$ . On the other hand, it can be extended to a holomorphic map in  $\widehat{\mathbb{C}} \setminus \{e^{i\theta} : \theta \in [0, \pi]\}$ . Hence  $Q$  can also be seen a discontinuous function defined on the whole Riemann sphere and the discontinuous points are contained in  $\{e^{i\theta} : \theta \in [0, \pi]\}$ . Therefore,  $Q$  has definition on  $\partial\mathbb{D}(1, \varepsilon_1)$  and on which one can calculate the maximum and minimum of  $|Q(\zeta)|$  etc. We will use this fact to do the estimations in the following.

Similarly, one can check that

$$\begin{aligned} \min_{\zeta \in \partial \mathbb{D}(-1, \varepsilon_2) \setminus \mathbb{D}} |Q(\zeta)| &\geq \min_{\theta \in [0, 2\pi]} |Q(-1 + \varepsilon_2 e^{i\theta})| (\doteq 7.3676 \dots \times 10^{-8}) \\ &> cv e^{-2\pi\eta} (\doteq 5.9124 \dots \times 10^{-8}). \end{aligned} \quad (3.15^*)$$

As above, for the estimation on  $\partial \mathbb{D}(c_-^Q, \varepsilon_3)$ , we have

$$\begin{aligned} \max_{\zeta \in \partial \mathbb{D}(c_-^Q, \varepsilon_3) \setminus \mathbb{D}} |Q(\zeta)| &\leq \max_{\theta \in [0, 2\pi]} |Q(c_-^Q + \varepsilon_3 e^{i\theta})| (\doteq 3.3960 \dots \times 10^9) \\ &< cv e^{2\pi\eta} (\doteq 4.8982 \dots \times 10^9). \end{aligned} \quad (3.16^*)$$

By the symmetry, the same estimation as (3.16\*) holds for  $\mathbb{D}(c_+^Q, \varepsilon_3)$ . Above all, it follows that  $\Upsilon^Q \setminus U_\eta^Q \subset \mathbb{D}(1, \varepsilon_1) \cup \mathbb{D}(-1, \varepsilon_2) \cup \mathbb{D}(c_+^Q, \varepsilon_3) \cup \mathbb{D}(c_-^Q, \varepsilon_3)$ .

(b) In order to prove  $\overline{\mathbb{D}}, \overline{\mathbb{D}}(1, \varepsilon_1), \overline{\mathbb{D}}(-1, \varepsilon_2), \overline{\mathbb{D}}(c_+^Q, \varepsilon_3)$  and  $\overline{\mathbb{D}}(c_-^Q, \varepsilon_3)$  are contained in  $\text{int } E$ , we only need to show that any one of them has no intersection with  $\partial E$ . For this, we first parameterize  $\partial E$  by  $x = x_E + a_E t$ ,  $y = \pm b_E \sqrt{1 - t^2}$  ( $-1 \leq t \leq 1$ ), where  $x_E = -0.053$ ,  $a_E = 1.057$  and  $b_E = 1.029$ . Define

$$\begin{aligned} h_1(t) &:= x^2 + y^2 - 1.006 = (a_E^2 - b_E^2)t^2 + 2a_E x_E t + b_E^2 + x_E^2 - 1.006; \text{ and} \\ h_2(t) &:= (x+1)^2 + y^2 - \varepsilon_2^2 = (a_E^2 - b_E^2)t^2 + 2a_E(x_E + 1)t + b_E^2 + x_E^2 - \varepsilon_2^2 + 2x_E + 1. \end{aligned}$$

We denote

$$t_1 = -\frac{a_E x_E}{a_E^2 - b_E^2} (\doteq 0.9591 \dots) \quad \text{and} \quad t_2 = -\frac{a_E(x_E + 1)}{a_E^2 - b_E^2} (\doteq -17.1377 \dots). \quad (3.17^*)$$

Note that  $a_E^2 - b_E^2 > 0$ , the quadratic polynomials  $h_1$  and  $h_2$  have minimum at  $t_1$  and  $t_2$  respectively. On the other hand, since  $t_1 \in [-1, 1]$  and  $t_2 < -1$ , the minimum of  $h_1$  and  $h_2$  within  $[-1, 1]$  will be attained by

$$h_1(t_1) (\doteq 0.0019 \dots) > 0 \quad \text{and} \quad h_2(-1) = 0.0021 > 0. \quad (3.18^*)$$

This means that  $\overline{\mathbb{D}}(0, 1.006)$  and  $\overline{\mathbb{D}}(-1, \varepsilon_2)$  have no intersection with  $\partial E$ . In particular, they are contained in  $\text{int } E$  since  $x_E - a_E < -(1 + \varepsilon_2) < -1.006 < x_E$ . Note that  $1 + \varepsilon_1 < 1 + \varepsilon_3 < 1.006$ . Hence  $\overline{\mathbb{D}}(1, \varepsilon_1) \subset \overline{\mathbb{D}}(0, 1.006)$  and  $\overline{\mathbb{D}}(c_\pm^Q, \varepsilon_3) \subset \overline{\mathbb{D}}(0, 1.006)$ . The proof is complete.  $\square$

**Lemma 3.22.** *Let  $a_4 = -0.24 + 0.64i$ ,  $a_5 = 0.88 + 0.24i$ ,  $\varepsilon_4 = 0.9$  and  $\varepsilon_5 = 0.5$ . Then*

- (a)  $\overline{\mathbb{D}}(a_4, \varepsilon_4) \setminus (\overline{\mathbb{D}} \cup \overline{U}_+^Q) \subset \mathcal{U}_{4+}$  and  $\overline{\mathbb{D}}(\bar{a}_4, \varepsilon_4) \setminus (\overline{\mathbb{D}} \cup \overline{U}_-^Q) \subset \mathcal{U}_{5-}$ .
- (b)  $\overline{\mathbb{D}}(a_5, \varepsilon_5) \cup \overline{\mathbb{D}}(\bar{a}_5, \varepsilon_5) \setminus \overline{\mathbb{D}} \subset \text{int } (\overline{U}_{2+} \cup \overline{U}_{3-} \cup \overline{U}'_{4-} \cup \overline{U}'_{5+})$ .

See Figure 8 and Figure 11.

*Proof.* (a) By the symmetry, we only need to prove the second statement. By Lemma 3.14, we have  $\partial \mathbb{D} \cap \partial \mathbb{D}(\bar{a}_4, \varepsilon_4) = \{\zeta_1 := \bar{a}_4 + \varepsilon_4 e^{i\theta_1}, \zeta_2 := \bar{a}_4 + \varepsilon_4 e^{i\theta_2}\}$  with  $\theta_1, \theta_2 \in [0, 2\pi)$ , where

$$(\cos \theta_1, \sin \theta_1) \doteq (-0.8331 \dots, 0.5530 \dots), \quad (\cos \theta_2, \sin \theta_2) \doteq (0.9913 \dots, -0.1311 \dots). \quad (3.19^*)$$

Then  $\theta_1 = \arctan(\frac{\sin \theta_1}{\cos \theta_1}) + \pi$  and  $\theta_2 = \arctan(\frac{\sin \theta_2}{\cos \theta_2}) + 2\pi$ . Moreover, we have

$$\text{Re } \zeta_1 (\doteq -0.9898 \dots) > -1 \quad \text{and} \quad \text{Re } \zeta_2 (\doteq 0.6522 \dots) < \text{Re } c_-^Q (\doteq 0.6611 \dots). \quad (3.20^*)$$

Therefore, we only need to prove that  $Q([\zeta_1, \zeta_2] \cap \partial \mathbb{D}(\bar{a}_4, \varepsilon_4)) \cap (-\infty, 0] = \emptyset$  since  $Q(\gamma_{c3}) = \Gamma_c^Q = (-\infty, 0)$ . This is true since

$$\min_{\zeta \in [\zeta_1, \zeta_2] \cap \partial \mathbb{D}(\bar{a}_4, \varepsilon_4)} \text{Re } \sqrt{Q(\zeta)} = \min_{\theta \in [\theta_1, \theta_2]} \text{Re } \sqrt{Q(\bar{a}_4 + \varepsilon_4 e^{i\theta})} (\doteq 9.5087 \dots \times 10^{-4}) > 0. \quad (3.21^*)$$

(b) Similarly, by the symmetry, it is sufficient to prove that  $\ell := (\partial \mathbb{D}(\bar{a}_5, \varepsilon_5) \setminus \mathbb{D}) \cap \mathbb{H}^-$  is contained in  $\mathcal{U}_{3-} \cup \mathcal{U}'_{5+} \cup \gamma_{c5}$ . Similar to the proof of (a), one can use Lemma 3.14 to



parameterize  $\ell$  such that its closure can be written as  $\bar{\ell} : \zeta = \bar{a}_5 + \varepsilon_5 e^{\theta\pi i} (\theta_3 \leq \theta \leq \theta_4)$ , where

$$\theta_3 (\doteq -0.6134\dots) \text{ and } \theta_4 (\doteq 0.1593\dots) \quad (3.22^*)$$

such that  $|\zeta_3| = 1$ ,  $\text{Im } \zeta_4 = 0$  and  $\zeta_3 := \bar{a}_5 + \varepsilon_5 e^{\theta_3\pi i}$ ,  $\zeta_4 := \bar{a}_5 + \varepsilon_5 e^{\theta_4\pi i}$ . Then we have

$$1 > \text{Re } \zeta_3 (\doteq 0.7056\dots) > \text{Re } c_-^Q (\doteq 0.6611\dots) \text{ and } 1 < \text{Re } \zeta_4 (\doteq 1.3186\dots) < cp. \quad (3.23^*)$$

This means that  $\zeta_3 \in (c_-^Q, 1)|_{\partial\mathbb{D}} \subset \partial\mathcal{U}'_{5+}$  and  $\zeta_4 \in \gamma_{b2} \subset \partial\mathcal{U}_{3-}$ . By

$$\min_{\theta \in [\theta_3, -0.55]} \text{Im } Q(\bar{a}_5 + \varepsilon_5 e^{\theta\pi i}) (\doteq 40.0808\dots) > 0, \quad (3.24^*)$$

we know that  $\ell' := \{\bar{a}_5 + \varepsilon_5 e^{\theta\pi i} : \theta_3 \leq \theta \leq -0.55\}$  is contained in  $\bar{\mathcal{U}}'_{5+}$ . On the other hand, we have

$$\min_{\theta \in [-0.55, 0.15]} \text{Re } \sqrt{-Q(\bar{a}_5 + \varepsilon_5 e^{\theta\pi i})} (\doteq 0.0665\dots) > 0. \quad (3.25^*)$$

This means that  $Q(\zeta) \cap [0, +\infty) = \emptyset$  if  $\zeta \in \ell'' := \{\bar{a}_5 + \varepsilon_5 e^{\theta\pi i} : -0.55 \leq \theta \leq 0.15\}$ . Note that  $(0, +\infty) = \Gamma_a^Q \cup \Gamma_b^Q$ . Hence  $\ell''$  is disjoint with any  $\gamma_{ai}$  or  $\gamma_{bi}$ . Since  $\bar{a}_5 + \varepsilon_5 e^{-0.55\pi i}$  is contained in  $\mathcal{U}'_{5+}$ , hence  $\ell''$  is contained in  $\mathcal{U}'_{5+} \cup \mathcal{U}_{3-} \cup \gamma_{c5}$ . By

$$Q(\zeta_4) (\doteq 1.2884\dots) \in \Gamma_b^Q \text{ and } r' := \max_{\theta \in [0.15, \theta_4]} |Q(\bar{a}_5 + \varepsilon_5 e^{\theta\pi i}) - Q(\zeta_4)| (\doteq 0.1735\dots), \quad (3.26^*)$$

it follows that  $\ell''' := \{\bar{a}_5 + \varepsilon_5 e^{\theta\pi i} : 0.15 \leq \theta \leq \theta_4\}$  is contained in  $\mathcal{U}_{3-} \cup \gamma_{b2}$  since  $\bar{a}_5 + \varepsilon_5 e^{0.15\pi i}$  is contained in  $\mathcal{U}_{3-}$  and  $Q(\zeta_4) + r' < cv$ ,  $Q(\zeta_4) - r' > 0$ . Above all, we have proved that  $\ell'$ ,  $\ell''$  and  $\ell'''$  are all contained in  $\bar{\mathcal{U}}'_{5+} \cup \bar{\mathcal{U}}_{3-}$ . Note that  $\bar{\ell} = \ell' \cup \ell'' \cup \ell'''$ . The proof is complete.  $\square$

**Lemma 3.23.** *The set  $\bar{U}_+^Q \cup \bar{U}_-^Q$  is contained in the interior of  $E$ .*

See Figure 10.

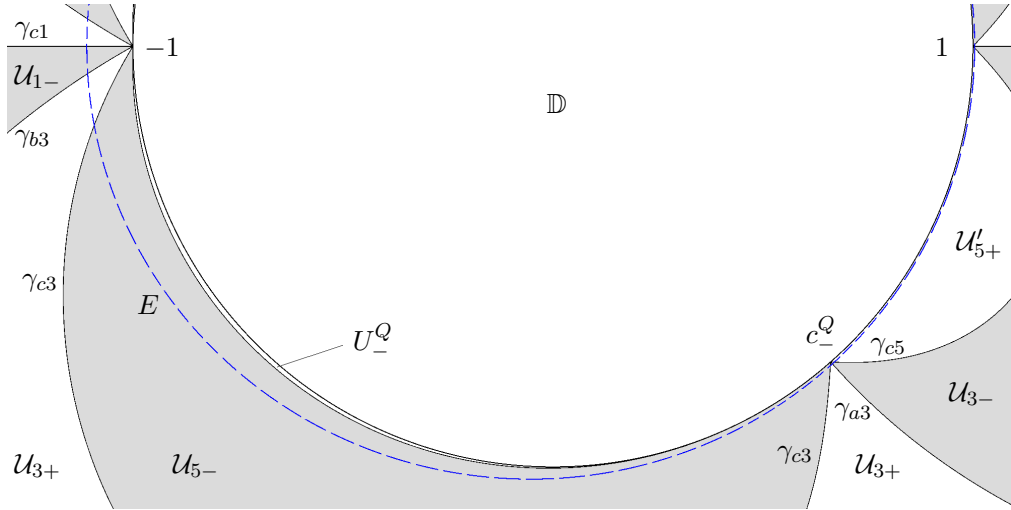


Figure 10: The ellipse  $E$  and  $U_-^Q$ . It can be seen clearly in this figure that  $U_-^Q$  is contained in the interior of  $E$ .

*Proof.* By the symmetry, it is sufficient to prove  $\bar{U}_-^Q \subset \text{int } E$ . We first parameterize the lower half part of  $\partial E$  by

$$\Gamma : y(x) = -b_E \sqrt{1 - (x - x_E)^2 / a_E^2}, \text{ where } x_E - a_E \leq x \leq x_E + a_E.$$

Let  $x_1 = x_E - a_E = -1.11$ ,  $x_2 = -1.109$ ,  $x_3 = -1.105$ ,  $x_4 = 0.664$ ,  $x_5 = 0.666$ ,  $x_6 = 0.8$  and denote  $z_i = x_i + y(x_i)i$  for  $1 \leq i \leq 6$ . Let  $\Gamma_i$  be the subarc of  $\Gamma = \partial E \cap \{\zeta : \text{Im } \zeta \leq 0\}$  with end points  $z_i$  and  $z_{i+1}$  for  $1 \leq i \leq 5$ .

We claim that (i)  $\text{Re } \sqrt{-Q(\zeta)} > 0$  for  $\zeta \in \Gamma_1$ ; (ii)  $0 < \text{Re } Q(\zeta) < cv$  for  $\zeta \in \Gamma_2$ ; (iii)  $\text{Re } \sqrt{-Q(\zeta)} > 0$  for  $\zeta \in \Gamma_3$ ; (iv)  $\text{Re } Q(\zeta) > cv$  for  $\zeta \in \Gamma_4$ ; (v)  $\text{Re } \sqrt{-Q(\zeta)} > 0$  for  $\zeta \in \Gamma_5$ ; and (vi)  $z_6 \in \mathbb{D}(\bar{a}_5, \varepsilon_5)$ . By (i), as analyzed after (3.25\*), it means that  $Q(\Gamma_1) \cap [0, +\infty) = \emptyset$ . Since  $(0, +\infty) = \Gamma_a^Q \cup \Gamma_b^Q$ , hence  $\Gamma_1$  is disjoint with any  $\gamma_{ai}$  or  $\gamma_{bi}$ . Because  $z_1 \in \gamma_{c1}$ , hence  $\Gamma_1 \subset \mathcal{U}_{1+} \cup \mathcal{U}_{1-} \cup \gamma_{c1}$ . Similarly, by (ii),  $\Gamma_2 \subset \mathcal{U}_{1-} \cup \mathcal{U}_{3+} \cup \gamma_{b3}$ . By (iii),  $\Gamma_3 \subset \mathcal{U}_{3+} \cup \mathcal{U}_{5-} \cup \gamma_{c3}$ . By (iv),  $\Gamma_4 \subset \mathcal{U}_{3+} \cup \mathcal{U}_{3-} \cup \gamma_{a3}$ . By (v),  $\Gamma_5 \subset \mathcal{U}_{3-} \cup \mathcal{U}_{5+}' \cup \gamma_{c5}$ . By (vi) and Lemma 3.22 (b), it follows that  $\bigcup_{i=1}^5 \Gamma_i \subset \text{int}(\bar{\mathcal{U}}_1 \cup \bar{\mathcal{U}}_3 \cup \mathcal{U}_{5-} \cup \mathcal{U}_{5+}')$ . On the other hand,  $\bar{U}_-^Q \cap \text{int}(\bar{\mathcal{U}}_1 \cup \bar{\mathcal{U}}_3 \cup \mathcal{U}_{5-} \cup \mathcal{U}_{5+}') = \emptyset$  and  $[-1, c_-^Q]_{\partial \mathbb{D}} \subset \bar{U}_-^Q$ . Therefore,  $\bar{U}_-^Q \subset \text{int } E$  since  $\bigcup_{i=1}^5 \Gamma_i \subset \Gamma$ . The claims (i)-(v) can be verified by the following numerical estimations:

$$\begin{aligned}
\min_{\zeta \in \Gamma_1} \text{Re } \sqrt{-Q(\zeta)} &= \min_{x \in [x_1, x_2]} \text{Re } \sqrt{-Q(x + y(x)i)} (\doteq 1.8855 \times 10^{-4}) > 0; \\
\min_{\zeta \in \Gamma_2} \text{Re } Q(\zeta) &= \min_{x \in [x_2, x_3]} \text{Re } Q(x + y(x)i) (\doteq 1.2551 \times 10^{-7}) > 0; \\
\max_{\zeta \in \Gamma_2} \text{Re } Q(\zeta) &= \max_{x \in [x_2, x_3]} \text{Re } Q(x + y(x)i) (\doteq 3.5679 \times 10^{-7}) < cv; \\
\min_{\zeta \in \Gamma_3} \text{Re } \sqrt{-Q(\zeta)} &= \min_{x \in [x_3, x_4]} \text{Re } \sqrt{-Q(x + y(x)i)} (\doteq 4.8927 \times 10^{-4}) > 0; \\
\min_{\zeta \in \Gamma_4} \text{Re } Q(\zeta) &= \max_{x \in [x_4, x_5]} \text{Re } Q(x + y(x)i) (\doteq 9.0269 \times 10^8) > cv; \\
\min_{\zeta \in \Gamma_5} \text{Re } \sqrt{-Q(\zeta)} &= \min_{x \in [x_5, x_6]} \text{Re } \sqrt{-Q(x + y(x)i)} (\doteq 2.0538 \dots) > 0.
\end{aligned} \tag{3.27*}$$

The claim that  $z_6 \in \mathbb{D}(\bar{a}_5, \varepsilon_5)$  is also true since

$$|z_6 - \bar{a}_5| = |x_6 + y(x_6)i - \bar{a}_5| (\doteq 0.3762 \dots) < \varepsilon_5 = 0.5. \tag{3.28*}$$

The proof is complete.  $\square$

*Proof of Proposition 3.2.* By Lemma 3.23, we have  $\widehat{\mathbb{C}} \setminus \text{int } E \subset \widehat{\mathbb{C}} \setminus (\bar{U}_+^Q \cup \bar{U}_-^Q)$ . By Lemma 3.21, we have

$$\widehat{\mathbb{C}} \setminus \text{int } E \subset \Upsilon^Q \setminus (\overline{\mathbb{D}}(1, \varepsilon_1) \cup \overline{\mathbb{D}}(-1, \varepsilon_2) \cup \overline{\mathbb{D}}(c_+^Q, \varepsilon_3) \cup \overline{\mathbb{D}}(c_-^Q, \varepsilon_3)) \subset U_\eta^Q \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

Therefore, we have

$$\bar{V} = \psi_1(\widehat{\mathbb{C}} \setminus \text{int } E) \subset \psi_1(U_\eta^Q) = U_\eta^P = V'.$$

This ends the proof of Proposition 3.2.  $\square$

Recall that  $\mathcal{U}_i = \mathcal{U}_{i+} \cup \mathcal{U}_{i-} \cup \gamma_{ai}$  for  $i = 1, 2, 3$ . We denote  $\mathcal{U}_{12}' = \mathcal{U}_1 \cup \mathcal{U}_{2-} \cup \mathcal{U}_{3+} \cup \gamma_{b1} \cup \gamma_{b3}$ .

**Lemma 3.24.** *Let  $R = 100$ ,  $\rho = 0.05$ ,  $a_6 = 0.71 + 0.89i$ ,  $a_7 = -1.45$ ,  $\varepsilon_6 = 0.41$ ,  $\varepsilon_7 = 1$  and  $r_1 = 1.25$ .*

- (a) *If  $\zeta \in \mathbb{C} \setminus \mathbb{D}$  and  $|\zeta - a_6| \leq \varepsilon_6$  (or  $|\zeta - \bar{a}_6| \leq \varepsilon_6$ ), then  $|Q(\zeta)| > R = 100$ .*
- (b) *If  $\zeta \in \mathbb{C} \setminus \mathbb{D}$  and  $|\zeta - a_7| \leq \varepsilon_7$ , then  $|Q(\zeta)| < \rho = 0.05$ .*
- (c) *The ellipse  $E_{r_1}$  is covered by the union of eight disks:  $\mathbb{D}$ ,  $\mathbb{D}(a_4, \varepsilon_4)$ ,  $\mathbb{D}(\bar{a}_4, \varepsilon_4)$ ,  $\mathbb{D}(a_5, \varepsilon_5)$ ,  $\mathbb{D}(\bar{a}_5, \varepsilon_5)$ ,  $\mathbb{D}(a_6, \varepsilon_6)$ ,  $\mathbb{D}(\bar{a}_6, \varepsilon_6)$  and  $\mathbb{D}(a_7, \varepsilon_7)$ . Hence*

$$\mathbb{C} \setminus \left( \mathbb{D} \cup \bigcup_{i=4}^7 \mathbb{D}(a_i, \varepsilon_i) \cup \bigcup_{i=4}^6 \mathbb{D}(\bar{a}_i, \varepsilon_i) \right) \subset \mathbb{C} \setminus E_{r_1}.$$

- (d) *If  $\zeta \in \bar{\mathcal{U}}_1$  and  $|Q(\zeta)| > \rho$ , then  $\zeta \in \mathbb{C} \setminus E_{r_1}$ . Moreover, if  $\zeta \in \mathcal{U}_{12}'$  and  $\rho \leq |Q(\zeta)| \leq R$ , then  $\zeta \in \mathbb{C} \setminus E_{r_1}$ .*

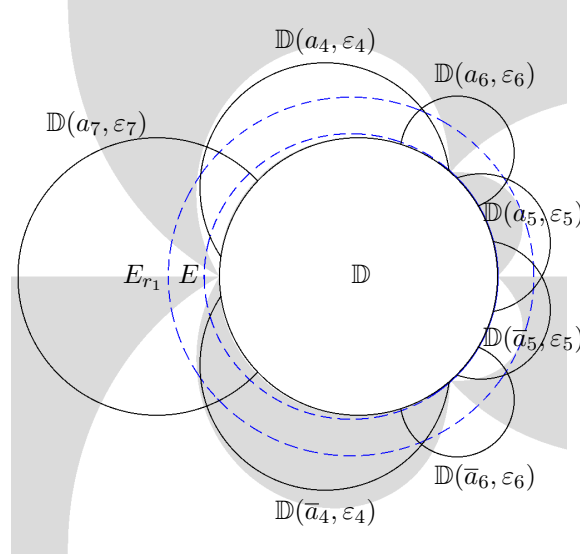


Figure 11: The ellipses  $E$ ,  $E_{r_1}$  and the eight disks appeared in Lemmas 3.22 and 3.24. It can be seen in this figure that the ellipse  $E_{r_1}$  is covered by the union of these eight disks.

See Figure 11.

*Proof.* (a) We consider the minimum of  $|Q(\zeta)|$  on the circle  $\partial\mathbb{D}(\bar{a}_6, \varepsilon_6)$  and have the following numerical estimate

$$\min_{\zeta \in \partial\mathbb{D}(\bar{a}_6, \varepsilon_6) \setminus \mathbb{D}} |Q(\zeta)| \geq \min_{\theta \in [0, 2\pi]} |Q(\bar{a}_6 + \varepsilon_6 e^{i\theta})| (\doteq 115.0061 \dots) > R = 100. \quad (3.29^*)$$

By the minimum modulus principle<sup>8</sup>, if  $\zeta \in \mathbb{C} \setminus \mathbb{D}$  and  $|\zeta - \bar{a}_6| \leq \varepsilon_6$ , then  $|Q(\zeta)| > R = 100$ . Since  $Q|_{\mathbb{C} \setminus \mathbb{D}}$  has the symmetric dynamics respect to the real axis, we have the same estimate as above for  $\partial\mathbb{D}(a_6, \varepsilon_6)$ .

(b) Consider the maximum of  $|Q(\zeta)|$  on the circle  $\partial\mathbb{D}(\bar{a}_7, \varepsilon_7)$  and we have the following numerical estimate

$$\max_{\zeta \in \partial\mathbb{D}(\bar{a}_7, \varepsilon_7) \setminus \mathbb{D}} |Q(\zeta)| \leq \max_{\theta \in [0, 2\pi]} |Q(a_7 + \varepsilon_7 e^{i\theta})| (\doteq 0.0469 \dots) < \rho = 0.05. \quad (3.30^*)$$

By the maximum modulus principle, if  $\zeta \in \mathbb{C} \setminus \mathbb{D}$  and  $|\zeta - a_7| \leq \varepsilon_7$ , then  $|Q(\zeta)| < \rho = 0.05$ .

(c) We need a useful criterion proved by Inou and Shishikura [IS08, Sublemma 5.18]: Let  $\Gamma = \{x + yi : (\frac{x}{a})^2 + (\frac{y}{b})^2 = 1, y \geq 0\}$  with  $a > b > 0$ . If two points  $z_1, z_2 \in \Gamma$  are contained in a disk  $\mathbb{D}(\zeta_0, r)$  with  $\text{Im } \zeta_0 \geq 0$ , then so is the subarc of  $\Gamma$  between  $z_1$  and  $z_2$ .

We will find some points in  $\partial E_{r_1}$  and prove that they are contained in the corresponding disks. Then the result follows by the criterion stated above. In particular, we just need to verify the upper half part of  $\partial E_{r_1}$  since the objects we consider are symmetric about the real axis. Note that  $\partial E_{r_1}$  is an ellipse and its upper half part can be parametrized by

$$\Gamma : y(x) = b_E(r_1) \sqrt{1 - \left( \frac{x - x_E}{a_E(r_1)} \right)^2}, \text{ where } x_E - a_E(r_1) \leq x \leq x_E + a_E(r_1).$$

Let  $x_1 = -1.05$ ,  $x_2 = 0.47$ ,  $x_3 = 1.05$  and denote  $z_1 = x_1 + y(x_1)i$ ,  $z_2 = x_2 + y(x_2)i$  and  $z_3 = x_3 + y(x_3)i$ . Then  $z_1$ ,  $z_2$  and  $z_3$  divide the curve  $\Gamma$  into four subarcs  $\Gamma_1$ ,  $\Gamma_2$ ,  $\Gamma_3$  and  $\Gamma_4$

<sup>8</sup>As stated before,  $Q$  can be seen as a holomorphic map in  $\widehat{\mathbb{C}} \setminus \{e^{i\theta} : \theta \in [0, \pi]\}$  and a discontinuous map on  $\widehat{\mathbb{C}}$ . Therefore,  $Q$  is holomorphic in a neighborhood of  $\mathbb{D}(\bar{a}_6, \varepsilon_6) \subset \mathbb{H}^-$ . Moreover,  $Q^{-1}(0) = \{1, -1\}$  (the preimage is taken in  $\widehat{\mathbb{C}}$ ). This means that  $0 \notin Q(\mathbb{D}(\bar{a}_6, \varepsilon_6))$  and hence we can apply the minimum modulus principle.

from left to right. The end points of  $\Gamma_1$ ,  $x_E - a_E(1.25) = -1.36795$  and  $z_1$ , are contained in  $\mathbb{D}(a_7, \varepsilon_7)$  since

$$|-1.36795 - a_7| < 1 = \varepsilon_7 \text{ and } (x_1 - a_7)^2 + y(x_1)^2 - \varepsilon_7^2 (\doteq -0.1297\dots) < 0. \quad (3.31^*)$$

The end points of  $\Gamma_2$ ,  $z_1$  and  $z_2$ , are contained in  $\mathbb{D}(a_4, \varepsilon_4)$ , since

$$\begin{aligned} (x_1 - \operatorname{Re} a_4)^2 + (y(x_1) - \operatorname{Im} a_4)^2 - \varepsilon_4^2 (\doteq -0.1127\dots) &< 0 \text{ and} \\ (x_2 - \operatorname{Re} a_4)^2 + (y(x_2) - \operatorname{Im} a_4)^2 - \varepsilon_4^2 (\doteq -0.0078\dots) &< 0. \end{aligned} \quad (3.32^*)$$

The end points of  $\Gamma_3$ ,  $z_2$  and  $z_3$ , are contained in  $\mathbb{D}(a_6, \varepsilon_6)$ , since

$$\begin{aligned} (x_2 - \operatorname{Re} a_6)^2 + (y(x_2) - \operatorname{Im} a_6)^2 - \varepsilon_6^2 (\doteq -0.0229\dots) &< 0 \text{ and} \\ (x_3 - \operatorname{Re} a_6)^2 + (y(x_3) - \operatorname{Im} a_6)^2 - \varepsilon_6^2 (\doteq -0.0177\dots) &< 0. \end{aligned} \quad (3.33^*)$$

The end points of  $\Gamma_4$ ,  $z_3$  and  $x_E + a_E(1.25) = 1.26195$ , are contained in  $\mathbb{D}(a_5, \varepsilon_5)$ , since

$$\begin{aligned} (x_3 - \operatorname{Re} a_5)^2 + (y(x_3) - \operatorname{Im} a_5)^2 - \varepsilon_5^2 (\doteq -0.0060\dots) &< 0 \text{ and} \\ |1.26195 - a_5| (\doteq 0.4510\dots) &< 0.5 = \varepsilon_5. \end{aligned} \quad (3.34^*)$$

Therefore the upper half part of  $\partial E_{r_1}$  is contained in  $\bigcup_{i=4}^7 \mathbb{D}(a_i, \varepsilon_i)$ . But it is not enough to prove (c). We need to prove that these disks have intersection points in the unit disk<sup>9</sup>. Actually, a direct calculation shows that

$$\begin{aligned} \partial \mathbb{D}(a_5, \varepsilon_5) \cap \mathbb{R} \cap \mathbb{D} &\neq \emptyset, \quad \partial \mathbb{D}(a_7, \varepsilon_7) \cap \mathbb{R} \cap \mathbb{D} \neq \emptyset, \quad \partial \mathbb{D}(a_7, \varepsilon_7) \cap \mathbb{D}(a_4, \varepsilon_4) \cap \mathbb{D} \neq \emptyset, \\ \partial \mathbb{D}(a_4, \varepsilon_4) \cap \mathbb{D}(a_6, \varepsilon_6) \cap \mathbb{D} &\neq \emptyset, \quad \partial \mathbb{D}(a_6, \varepsilon_6) \cap \mathbb{D}(a_5, \varepsilon_5) \cap \mathbb{D} \neq \emptyset. \end{aligned} \quad (3.35^*)$$

Now we can conclude that  $E_{r_1}$  is contained in the union of the eight disks stated in the Lemma.

(d) Suppose  $\zeta \in \overline{\mathcal{U}}_1$ . Then  $\zeta \notin \overline{\mathbb{D}}(a_4, \varepsilon_4) \cup \overline{\mathbb{D}}(\overline{a}_4, \varepsilon_4) \cup \overline{\mathbb{D}}(a_5, \varepsilon_5) \cup \overline{\mathbb{D}}(\overline{a}_5, \varepsilon_5)$  by Lemma 3.22. By (a), we know that  $\overline{\mathbb{D}}(a_6, \varepsilon_6) \cup \overline{\mathbb{D}}(\overline{a}_6, \varepsilon_6) \setminus \overline{\mathbb{D}}$  is contained in  $Q^{-1}(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}(0, R))$ . Note that  $Q : \mathcal{U}_1 \cup \gamma_{c1} \rightarrow \mathbb{C} \setminus [0, cv]$  is conformal. This means that  $\overline{\mathbb{D}}(a_6, \varepsilon_6) \cup \overline{\mathbb{D}}(\overline{a}_6, \varepsilon_6)$  must be contained in  $\mathbb{C} \setminus \overline{\mathcal{U}}_1$ . According to (b) and (c), we have  $\zeta \in \mathbb{C} \setminus E_{r_1}$ .

For the second statement, assume that  $\zeta \in \mathcal{U}'_{12}$  and  $\rho \leq |Q(\zeta)| \leq R$ . By Lemmas 3.22, we know that  $\zeta \notin \mathbb{D}(a_4, \varepsilon_4) \cup \mathbb{D}(\overline{a}_4, \varepsilon_4) \cup \mathbb{D}(a_5, \varepsilon_5) \cup \mathbb{D}(\overline{a}_5, \varepsilon_5)$ . By (a) and (b),  $\zeta$  is not contained in  $\mathbb{D}(a_6, \varepsilon_6) \cup \mathbb{D}(\overline{a}_6, \varepsilon_6) \cup \mathbb{D}(a_7, \varepsilon_7)$ . By (c), we have  $\zeta \in \mathbb{C} \setminus E_{r_1}$ . The proof is complete.  $\square$

### 3.6. Estimates on $Q$ : Part II.

**Lemma 3.25.** *We can write*

$$Q(\zeta) = \zeta + b_0 + \frac{b_1}{\zeta} + Q_2(\zeta),$$

where

$$b_0 = \frac{4\sqrt{5}\tau}{3} + \frac{2\nu\sigma(2-\sigma)(\kappa^2(\sigma-2)+\sigma)}{\tau(\sigma(\kappa^2+1)-2)} (\doteq 7.3476\dots), \text{ and} \quad (3.36^*)$$

$$\begin{aligned} b_1 &= b_0^2 - \frac{38\tau^2}{9} - \frac{2\nu\sigma(2-\sigma)}{3\tau(\sigma(\kappa^2+1)-2)} (8\sqrt{5}\sigma\tau - 2\sigma^2 - 1 + (8\sqrt{5}(\sigma-2)\tau - 2\sigma^2 + 8\sigma - 9)\kappa^2) \\ &(\doteq 21.4270\dots) \end{aligned} \quad (3.37^*)$$

<sup>9</sup>Inou and Shishikura don't need to do this because they chose two "nice" disks and it is obvious that the union of them cover the unit disk.

are two constants such that  $Q_2(\zeta) = \mathcal{O}(1/\zeta^2)$  as  $\zeta$  tends to  $\infty$ . For  $|\zeta| \geq r > 1$ , we have<sup>10</sup>

$$|Q_2(\zeta)| \leq Q_{2,max}(r) := \max_{\zeta \in \mathbb{C}: |\zeta|=r} \left| Q(\zeta) - \zeta - b_0 - \frac{b_1}{\zeta} \right|.$$

*Proof.* After expanding the polynomial  $P$ , we have

$$P(z) = z + \frac{4\sqrt{5}}{3}z^2 + \frac{38}{9}z^3 + \frac{4\sqrt{5}}{3}z^4 + z^5.$$

Consider the conjugacy  $\widehat{Q}(z) = \psi_0 \circ Q \circ \psi_0^{-1}$  as in the proof of Lemma 3.19. By (3.9), we have

$$\psi''_{1,2}(z) = \frac{\mu}{z^{2+\sigma}}(1 + \sigma + (1 - \sigma)z^2) \text{ and } \widetilde{\psi}''_{1,1}(z) = \frac{4\tau\kappa i}{(\tau - z)^3}. \quad (3.38)$$

Therefore, by (3.7) and (3.8), we have

$$\begin{aligned} \widehat{Q}''(0) &= P''(0) \cdot [\psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}'_{1,1}(0)]^2 + P'(0) \cdot \psi''_{1,2}(\kappa i) \cdot [\widetilde{\psi}'_{1,1}(0)]^2 + P'(0) \cdot \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}''_{1,1}(0) \\ &= \frac{8\sqrt{5}}{3} \cdot 1^2 + 1 \cdot \frac{\mu}{(\kappa i)^{2+\sigma}}(1 + \sigma - (1 - \sigma)\kappa^2) \cdot \left(\frac{2\kappa i}{\tau}\right)^2 + 1 \cdot \frac{\mu}{(\kappa i)^{1+\sigma}}(-\kappa^2 - 1) \cdot \frac{4\kappa i}{\tau^2} \\ &= \frac{8\sqrt{5}}{3} + \frac{\mu}{(\kappa i)^\sigma} \cdot \frac{4(\kappa^2(\sigma - 2) + \sigma)}{\tau^2} = \frac{8\sqrt{5}}{3} + \frac{4\nu\sigma(2 - \sigma)(\kappa^2(\sigma - 2) + \sigma)}{\tau^2(\sigma(\kappa^2 + 1) - 2)}. \end{aligned}$$

By (3.38), we have

$$\psi'''_{1,2}(z) = \frac{\mu}{z^{3+\sigma}}(-2 - (z^2 + 3)\sigma + (z^2 - 1)\sigma^2) \text{ and } \widetilde{\psi}'''_{1,1}(z) = \frac{12\tau\kappa i}{(\tau - z)^4}.$$

Therefore, we have

$$\begin{aligned} \widehat{Q}'''(0) &= P'''(0) \cdot 1^3 + 2P''(0) \cdot \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}'_{1,1}(0) \cdot (\psi''_{1,2}(\kappa i) \cdot [\widetilde{\psi}'_{1,1}(0)]^2 + \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}''_{1,1}(0)) \\ &\quad + P''(0) \cdot \psi''_{1,2}(\kappa i) \cdot [\widetilde{\psi}'_{1,1}(0)]^2 + \psi'''_{1,2}(\kappa i) \cdot [\widetilde{\psi}'_{1,1}(0)]^3 + 2\psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}'_{1,1}(0) \cdot \widetilde{\psi}''_{1,1}(0) \\ &\quad + P''(0) \cdot \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}''_{1,1}(0) + \psi''_{1,2}(\kappa i) \cdot \widetilde{\psi}'_{1,1}(0) \cdot \widetilde{\psi}''_{1,1}(0) + \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}'''_{1,1}(0) \\ &= P'''(0) + 3P''(0) \cdot (\psi''_{1,2}(\kappa i) \cdot [\widetilde{\psi}'_{1,1}(0)]^2 + \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}''_{1,1}(0)) + \psi'''_{1,2}(\kappa i) \cdot [\widetilde{\psi}'_{1,1}(0)]^3 \\ &\quad + 3\psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}'_{1,1}(0) \cdot \widetilde{\psi}''_{1,1}(0) + \psi'_{1,2}(\kappa i) \cdot \widetilde{\psi}'''_{1,1}(0) \\ &= \frac{76}{3} + \frac{\mu}{(\kappa i)^\sigma} \cdot \frac{4}{\tau^3} \cdot (8\sqrt{5}\sigma\tau - 2\sigma^2 - 1 + (8\sqrt{5}(\sigma - 2)\tau - 2\sigma^2 + 8\sigma - 9)\kappa^2) \\ &= \frac{76}{3} + \frac{4\nu\sigma(2 - \sigma)}{\tau^3(\sigma(\kappa^2 + 1) - 2)}(8\sqrt{5}\sigma\tau - 2\sigma^2 - 1 + (8\sqrt{5}(\sigma - 2)\tau - 2\sigma^2 + 8\sigma - 9)\kappa^2). \end{aligned}$$

Note that

$$\widehat{Q}(z) = z + \frac{\widehat{Q}''(0)}{2}z^2 + \frac{\widehat{Q}'''(0)}{6}z^3 + \mathcal{O}(z^4).$$

By a direct calculation, we have

$$Q(\zeta) = -\frac{\tau}{\widehat{Q}(-\frac{\tau}{\zeta})} = \zeta + \frac{\widehat{Q}''(0)\tau}{2} + \left( \frac{(\widehat{Q}''(0))^2}{4} - \frac{\widehat{Q}'''(0)}{6} \right) \frac{\tau^2}{\zeta} + \mathcal{O}\left(\frac{1}{\zeta^2}\right).$$

Then  $b_0$  and  $b_1$  can be calculated directly. The estimation on  $|Q_2(\zeta)|$  follows from the maximum modulus principle.  $\square$

**Lemma 3.26.**  $Q(\overline{\mathbb{V}}(13, \frac{\pi}{5})) \subset \mathbb{V}(19, \frac{\pi}{5}) \subset \mathbb{V}(cv, \frac{\pi}{5}).$

<sup>10</sup>In [IS08, Lemma 5.19], the corresponding  $Q_{2,max}$  can be written as a rational map with real coefficients, which is easy to calculate even by hand. However, in our setting, because of the complexity of the Riemann mapping  $\psi_{1,2}$ , the value of  $Q_{2,max}(r)$  for  $r > 1$  can only be calculated by the help of computer.

*Proof.* Suppose that  $\zeta \in \overline{\mathbb{V}}(13, \frac{\pi}{5})$  and let  $\zeta' = \zeta + 6$ . Since  $\zeta' \in \overline{\mathbb{V}}(19, \frac{\pi}{5})$ , hence it is sufficient to show that  $|\arg(Q(\zeta) - \zeta')| = |\arg(\frac{b_1}{\zeta} + b_0 - 7 + (1 + Q_2(\zeta)))| < \frac{\pi}{5}$ . Since  $\zeta \in \overline{\mathbb{V}}(13, \frac{\pi}{5})$ , then  $|\arg \zeta| < \frac{\pi}{5}$  and

$$|\arg(\frac{b_1}{\zeta} + b_0 - 7)| \leq |\arg b_1| + |\arg \zeta| + |\arg(b_0 - 7)| = |\arg \zeta| < \frac{\pi}{5}.$$

On the other hand, by Lemma 3.9 (a) and Lemma 3.25, we have

$$|\arg(1 + Q_2(\zeta))| \leq \arcsin Q_{2,max}(13) (\doteq 0.1499 \dots) < \frac{\pi}{5}. \quad (3.39^*)$$

Combine these two inequalities, we obtain that  $|\arg(Q(\zeta) - \zeta')| < \frac{\pi}{5}$ . Therefore, we have  $Q(\overline{\mathbb{V}}(13, \frac{\pi}{5})) \subset \mathbb{V}(19, \frac{\pi}{5}) \subset \mathbb{V}(cv, \frac{\pi}{5})$ .  $\square$

**Lemma 3.27.** (a) If  $|\zeta| \geq r > cp (\doteq 4.0843 \dots)$ , then

$$|\log Q'(\zeta)| \leq \text{Log} DQ_{max}(r) := \max_{\zeta \in \mathbb{C}: |\zeta|=r} |\log Q'(\zeta)|.$$

(b) If  $|\zeta| \geq 4.8$ , then  $\text{Re } Q'(\zeta) > 0$ . For any  $\theta \in \mathbb{R}$ ,  $Q$  is injective in  $\{\zeta : \text{Re}(\zeta e^{-i\theta}) > 4.8\}$ .

*Proof.* (a) This is an immediate corollary of the maximum modulus principle.

(b) By an numerical calculation, we have

$$\min_{\zeta: |\zeta|=4.8} \text{Re } Q'(\zeta) = \min_{\theta \in [0, 2\pi]} \text{Re } Q'(4.8e^{i\theta}) (\doteq 0.0292 \dots) > 0. \quad (3.40^*)$$

Let  $\zeta_0$  and  $\zeta_1$  be two different points in  $\{\zeta : \text{Re}(\zeta e^{-i\theta}) > 4.8\}$ . Then  $\zeta_0$  and  $\zeta_1$  can be connected by a segment in  $\{\zeta : \text{Re}(\zeta e^{-i\theta}) > 4.8\}$ . By (3.3), we have  $\text{Re} \frac{Q(\zeta_1) - Q(\zeta_0)}{\zeta_1 - \zeta_0} > 0$  since  $\text{Re } Q'(\zeta) > 0$ . Hence  $Q(\zeta_0) \neq Q(\zeta_1)$ . This means that  $Q$  is a injection in  $\{\zeta : \text{Re}(\zeta e^{-i\theta}) > 4.8\}$ .  $\square$

**3.7. Estimates on  $\varphi$ .** In the following, we always suppose that  $F = Q \circ \varphi^{-1} \in \mathcal{F}_1^Q$  if there is no special instruction. Then  $\varphi : \widehat{\mathbb{C}} \setminus E \rightarrow \widehat{\mathbb{C}} \setminus \{0\}$  is a normalized univalent mapping. Recall that  $\zeta(w)$  is a conformal mapping defined from  $\mathbb{C} \setminus \overline{\mathbb{D}}$  onto  $\mathbb{C} \setminus E$  in Lemma 3.10. The following lemma gives the various estimations on the univalent map  $\varphi$ . A proof of this lemma can be found in [IS08, §5.G].

**Lemma 3.28.** Suppose  $\varphi : \widehat{\mathbb{C}} \setminus E \rightarrow \widehat{\mathbb{C}} \setminus \{0\}$  is a normalized univalent map, i.e.  $\varphi(\infty) = \infty$  and  $\lim_{\zeta \rightarrow \infty} \frac{\varphi(\zeta)}{\zeta} = 1$ . Hence it can be written as

$$\varphi(\zeta) = \zeta + c_0 + \varphi_1(\zeta)$$

with  $c_0 \in \mathbb{C}$  and  $\lim_{\zeta \rightarrow \infty} \varphi_1(\zeta) = 0$ . Then the following estimations hold:

- (a)  $|c_0 - c_{00}| \leq c_{01,max}$ , where  $c_{00} := 0.053 = -x_E$ ,  $c_{01,max} := 2.086 = 2e_1$ .
- (b)  $\text{Image}(\varphi) \supset \{z : |z - (c_0 + x_E)| > 2e_1\} \supset \{z : |z| > 4e_1 = 4.172\}$ .
- (c)  $e_1|w| \left(1 - \frac{1}{|w|}\right)^2 \leq |\varphi(\zeta(w))| \leq e_1|w| \left(1 - \frac{1}{|w|}\right)^2$  for  $|w| > 1$ .
- (d)  $\left|\arg \frac{\varphi(\zeta(w))}{w}\right| \leq \log \frac{|w|+1}{|w|-1}$  for  $|w| > 1$ .
- (e)  $|\varphi_1(\zeta)| \leq \varphi_{1,max}(r) := a_E \sqrt{-\log \left(1 - \left(\frac{a_E}{r - |x_E|}\right)^2\right)}$  for  $|\zeta| \geq r > a_E + |x_E| = 1.11$ .
- (f)  $|\log \varphi'(z)| \leq \text{Log} D\varphi_{max}(r) := -\log \left(1 - \left(\frac{a_E}{r - |x_E|}\right)^2\right)$  for  $|\zeta| \geq r > a_E + |x_E| = 1.11$ .

**Lemma 3.29.** If  $\zeta \in \mathbb{C} \setminus \text{int } E_{r_1}$ , then  $|\varphi(\zeta)| > \rho$  and  $\left|\arg \frac{\varphi(\zeta)}{\zeta}\right| < \pi$ .

*Proof.* Let  $\zeta \in \mathbb{C} \setminus \text{int } E_{r_1}$ . By Lemma 3.10, one can write  $\zeta = \zeta(w)$  with  $|w| \geq r_1 = 1.25$ . By Lemma 3.28 (c), we have

$$|\varphi(\zeta)| = |\varphi(\zeta(w))| \geq e_1 |w| \left(1 - \frac{1}{|w|}\right)^2 \geq 1.043 \times 1.25 \left(1 - \frac{1}{1.25}\right)^2 = 0.05215 > \rho = 0.05.$$

By Lemma 3.28 (d), we have

$$\left| \arg \frac{\varphi(\zeta(w))}{w} \right| \leq \log \frac{1.25 + 1}{1.25 - 1} = 2 \log 3.$$

On the other hand, by Lemma 3.9 (a), we have

$$\left| \arg \frac{\zeta(w)}{w} \right| = \left| \arg \left( 1 + \frac{x_E}{e_1 w} + \frac{e_{-1}}{e_1 w^2} \right) \right| \leq \arcsin(|x_E| + |e_{-1}|) = \arcsin(0.067).$$

Therefore, we have

$$\left| \arg \frac{\varphi(\zeta)}{\zeta} \right| \leq \left| \arg \frac{\varphi(\zeta(w))}{w} \right| + \left| \arg \frac{\zeta(w)}{w} \right| \leq 2 \log 3 + \arcsin(0.067) (\doteq 2.2642 \dots) < \pi. \quad (3.41^*)$$

This ends the proof of Lemma 3.29.  $\square$

The following lemma will be used in §3.12.

**Lemma 3.30.** *Let*

$$\begin{aligned} \Omega_1 &= \{w = re^{i\theta} : r \geq r_1 = 1.25 \text{ and } |\theta| \leq 0.3\pi\}, \\ \Omega_2 &= \{w = re^{i\theta} : r \geq r_2 = 1.4 \text{ and } 0.3\pi \leq |\theta| \leq 0.4\pi\}, \text{ and} \\ \Omega_3 &= \{w = re^{i\theta} : r \geq r_3 = 1.54 \text{ and } 0.4\pi \leq |\theta| \leq \frac{\pi}{2}\}. \end{aligned}$$

*If  $\zeta \in \zeta(\Omega_1 \cup \Omega_2 \cup \Omega_3)$ , then  $\varphi(\zeta) \notin \mathbb{R}_-$ .*

*Proof.* By Lemma 3.28 (d), we know that for  $|w| > 1$ , then

$$|\arg \varphi(\zeta(w))| \leq |\arg w| + \left| \arg \frac{\varphi(\zeta(w))}{w} \right| \leq |\arg w| + \log \frac{|w| + 1}{|w| - 1}. \quad (3.42)$$

In order to prove the lemma, we only need to prove the following:

$$\text{if } w = re^{i\theta} \in \Omega_1 \cup \Omega_2 \cup \Omega_3, \text{ then } |\theta| + \log \frac{r + 1}{r - 1} < \pi.$$

However this has been verified by Inou and Shishikura in the proof of [IS08, Lemma 5.24]<sup>11</sup>.  $\square$

**3.8. Lifting  $Q$  and  $\varphi$  to  $X$ .** Recall that  $\mathcal{U}'_{12} = \mathcal{U}_1 \cup \mathcal{U}_{2-} \cup \mathcal{U}_{3+} \cup \gamma_{b1} \cup \gamma_{b3}$ .

**Definition.** Denote  $Y_{j\pm} = (Q|_{\bar{\mathcal{U}}_{j\pm}})^{-1}(\pi_X(X_{j\pm}))$  for  $j = 1, 2$ , except  $Y_{2+}$  is defined by<sup>12</sup>  $Y_{2+} = (Q|_{\bar{\mathcal{U}}_{3+}})^{-1}(\pi_X(X_{2+}))$ . Let

$$Y = Y_{1+} \cup Y_{1-} \cup Y_{2+} \cup Y_{2-},$$

which is subset of  $\mathcal{U}'_{12} \cup \mathbb{R}_- \subset \mathbb{C}$ . Define  $\tilde{Q} : Y \rightarrow X$  (whose well-definedness will be verified) by

$$\tilde{Q}(\zeta) = (\pi|_{X_{j\pm}})^{-1}(Q(\zeta)) \in X_{j\pm} \text{ for } \zeta \in Y_{j\pm}.$$

<sup>11</sup>In [IS08, Lemma 5.24], Inou and Shishikura proved that if  $\zeta \in \zeta(\{w = re^{i\theta} : r \geq r_1 = 1.25 \text{ and } |\theta| \leq \pi/2\})$ , then either  $\varphi(\zeta) \notin \mathbb{R}_-$  or  $\zeta$  is contained in the union of two disks. In particular, they proved that if  $\zeta \in \zeta(\Omega_1 \cup \Omega_2 \cup \Omega_3)$ , then  $\varphi(\zeta) \notin \mathbb{R}_-$ . In our case, we cannot find the corresponding disks. However, these two disks are not necessary in our proof since we can control the shape of  $Q^{-1}(W_0) \cap \mathcal{U}_{123}$  in Lemma 3.36.

<sup>12</sup>Compare the left picture in Figure 8 and the right picture in Figure 3. Although the idea of the definition is the same as that in [IS08], but the nominate is different.

We define

$$\tilde{Y} = \mathbb{C} \setminus (E_{r_1} \cup \mathbb{R}_+ \cup \overline{V}(13, \frac{\pi}{5})).$$

**Lemma 3.31.**  $Y \subset \tilde{Y}$ .

*Proof.* Assume that  $\zeta \in Y$ . If  $\zeta \in Y_{1\pm}(\subset \overline{U}_1)$ , then  $|Q(\zeta)| > \rho$ . If  $\zeta \in Y_{2\pm}$ , then  $\rho < |Q(\zeta)| < R$ . Therefore, in either case, we have  $\zeta \in \mathbb{C} \setminus E_{r_1}$  by Lemma 3.24 (d). Since  $\pi_X(X) \cap \overline{V}(cv, \frac{\pi}{5}) = \emptyset$ , we have  $Q(\zeta) \notin \overline{V}(cv, \frac{\pi}{5})$ . This means that  $\zeta \notin \overline{V}(13, \frac{\pi}{5})$  by Lemma 3.26.  $\square$

*Proof of Proposition 3.4.* (a) The proof is almost the same as that in [IS08, §5.H]. One only needs to check the consistency along the boundaries of  $Y_{j\pm}$  since  $\tilde{Q}$  maps  $Y_{j\pm}$  homeomorphically onto  $X_{j\pm}$  for  $j = 1, 2$ . We omit the details here.

(b) We prove that  $\varphi|_{\tilde{Y}}$  can be lifted to  $\tilde{\varphi}: \tilde{Y} \rightarrow X$  which is well-defined and holomorphic. Since  $Y \subset \tilde{Y}$  by Lemma 3.31, then Proposition 3.4 (b) will hold. By Lemma 3.28, if  $|\zeta| \geq 13$ , then we have

$$\begin{aligned} |\varphi(\zeta) - \zeta| &= |c_{00} + (c_0 - c_{00}) + \varphi_1(\zeta)| \leq c_{00} + c_{01, \max} + \varphi_{1, \max}(13) (\doteq 2.2254 \dots) \\ &< (cv - 13) \sin \frac{\pi}{5} (\doteq 2.3616 \dots). \end{aligned} \quad (3.43^*)$$

Therefore, if  $\zeta \in \mathbb{C} \setminus \overline{V}(13, \frac{\pi}{5})$  (in particular if  $\zeta \in \tilde{Y}$ ), then  $\varphi(\zeta) \notin \overline{V}(cv, \frac{\pi}{5})$  since the distance between  $\partial \overline{V}(13, \frac{\pi}{5})$  and  $\overline{V}(cv, \frac{\pi}{5})$  is  $(cv - 13) \sin \frac{\pi}{5}$ . On the other hand, if  $\zeta \in \mathbb{C} \setminus E$  and  $|\zeta| \leq 50$ , then

$$|\varphi(\zeta)| \leq 50 + c_{00} + c_{01, \max} + \varphi_{1, \max}(50) < 50 + 3 = 53. \quad (3.44^*)$$

Let  $\zeta \in \tilde{Y}$ . By Lemma 3.29, we have  $\left| \arg \frac{\varphi(\zeta)}{\zeta} \right| < \pi$  and  $|\varphi(\zeta)| > \rho$  for  $\zeta \in \tilde{Y}$  since  $\tilde{Y} \subset \mathbb{C} \setminus E_{r_1}$ . As in [IS08, §5.H], we define  $\tilde{\varphi}(\zeta) \in X$  such that  $\pi_X(\tilde{\varphi}(\zeta)) = \varphi(\zeta)$  and

$$\begin{aligned} \tilde{\varphi}(\zeta) &\in X_{1+} \cup X_{2-} \text{ if } \operatorname{Im} \zeta \geq 0 \text{ and } -\pi < \arg \frac{\varphi(\zeta)}{\zeta} \leq 0; \\ \tilde{\varphi}(\zeta) &\in X_{1-} \cup X_{2+} \text{ if } \operatorname{Im} \zeta < 0 \text{ and } 0 \leq \arg \frac{\varphi(\zeta)}{\zeta} < \pi; \\ \tilde{\varphi}(\zeta) &\in X_{1+} \cup X_{1-} \text{ otherwise.} \end{aligned}$$

We first prove that  $\tilde{\varphi}$  is well-defined. For this, we just need to show that if  $\tilde{\varphi}(\zeta)$  is defined to be in  $X_{2\pm}$ , then  $|\varphi(\zeta)| < R$  for the first two cases above. For the first case,  $\operatorname{Im} \zeta \geq 0$  and  $\varphi(\zeta)$  is contained in a half plane  $H = \{w : \arg \zeta - \pi < \arg w < \arg \zeta\}$ . Note that  $\zeta \in \mathbb{C} \setminus \overline{V}(13, \frac{\pi}{5})$ . If further  $|\zeta| \geq 50$ , then  $\varphi(\zeta) \notin \overline{V}(cv, \frac{\pi}{5})$  by Lemma 3.26 and  $|\varphi(\zeta) - \zeta| < 3$ ,  $|\varphi(\zeta)| \geq 47$  by (3.43\*). But  $\varphi(\zeta) \in H$ , hence  $\operatorname{Im} \varphi(\zeta) > 0$  since the distance between the two points in  $\mathbb{D}(0, 47) \cap \partial \overline{V}(cv, \frac{\pi}{5})$  is larger than 3. This means that  $\tilde{\varphi}(\zeta)$  should be defined in  $X_{1+}$ . On the other hand, if  $|\zeta| < 50$ , then  $|\varphi(\zeta)| < 53 < R = 100$  by (3.44\*). Therefore,  $\tilde{\varphi}$  is well-defined for the first case. Similar argument can be applied to the second case. The continuity of  $\tilde{\varphi}$  can be verified as in [IS08, §5.H]. Once the continuity of  $\tilde{\varphi}$  is obtained, then it is holomorphic.  $\square$

### 3.9. Estimates on $F$ .

**Lemma 3.32.** Suppose  $r \geq 5.4$ ,  $\theta \in \mathbb{R}$  and  $\operatorname{Re}(\zeta e^{-i\theta}) > r$ . Then the following estimates hold for  $z = \varphi(\zeta)$ :

(a)  $F(z) - z \in \mathbb{D}(b_0 - c_{00} + \frac{b_1 e^{-i\theta}}{2r}, \beta_{\max}(r))$ , where

$$\beta_{\max}(r) := c_{01, \max} + \frac{b_1}{2r} + Q_{2, \max}(r) + \varphi_{1, \max}(r);$$



(b)  $\text{Arg}\Delta F_{\min}(r, \theta) \leq \arg(F(z) - z) \leq \text{Arg}\Delta F_{\max}(r, \theta)$ , where

$$\begin{aligned} \text{Arg}\Delta F_{\{\min\}^{\max}}(r, \theta) := & -\arctan\left(\frac{\frac{b_1 \sin \theta}{2r}}{b_0 - c_{00} + \frac{b_1 \cos \theta}{2r}}\right) \\ & \pm \arcsin\left(\frac{\beta_{\max}(r)}{\sqrt{(b_0 - c_{00})^2 + \left(\frac{b_1}{2r}\right)^2 + 2(b_0 - c_{00})\left(\frac{b_1}{2r}\right)\cos \theta}}\right); \end{aligned}$$

(c)  $\text{Abs}\Delta F_{\min}(r, \theta) \leq |F(z) - z| \leq \text{Abs}\Delta F_{\max}(r, \theta)$ , where

$$\text{Abs}\Delta F_{\{\min\}^{\max}}(r, \theta) := \sqrt{(b_0 - c_{00})^2 + \left(\frac{b_1}{2r}\right)^2 + 2(b_0 - c_{00})\left(\frac{b_1}{2r}\right)\cos \theta} \pm \beta_{\max}(r);$$

(d)  $|\log F'(z)| \leq \text{Log}DF_{\max}(r) := \text{Log}DQ_{\max}(r) + \text{Log}D\varphi_{\max}(r)$ , where

$$\text{Log}DQ_{\max}(r) := \max_{\zeta: |\zeta|=r} |\log Q'(\zeta)|.$$

*Proof.* The proof can be obtained by an almost word for word copy from [IS08, Lemma 5.27]. Therefore, we will not include a proof here. If  $r \geq 5.4$ , we have

$$b_0 - c_{00} - \frac{b_1}{2 \times 5.4} (\doteq 5.3106 \dots) > \beta_{\max}(5.4) (\doteq 5.2667 \dots). \quad (3.45^*)$$

This is the unique requirement in the proof of [IS08, Lemma 5.27].  $\square$

**Lemma 3.33.**  $F(\overline{\mathbb{V}}(cv, \frac{\pi}{5})) \subset \mathbb{V}(19, \frac{\pi}{5}) \subset \mathbb{V}(cv, \frac{\pi}{5})$ .

*Proof.* By the estimate (3.43\*), if  $\zeta \in \mathbb{C} \setminus \overline{\mathbb{V}}(13, \frac{\pi}{5})$ , then  $\varphi(\zeta) \notin \overline{\mathbb{V}}(cv, \frac{\pi}{5})$ . This means that  $\varphi^{-1}(\overline{\mathbb{V}}(cv, \frac{\pi}{5})) \subset \overline{\mathbb{V}}(13, \frac{\pi}{5})$ . By Lemma 3.26, we have  $F(\overline{\mathbb{V}}(cv, \frac{\pi}{5})) = Q \circ \varphi^{-1}(\overline{\mathbb{V}}(cv, \frac{\pi}{5})) \subset Q(\overline{\mathbb{V}}(13, \frac{\pi}{5})) \subset \mathbb{V}(19, \frac{\pi}{5}) \subset \mathbb{V}(cv, \frac{\pi}{5})$ .  $\square$

### 3.10. Repelling Fatou coordinate $\tilde{\Phi}_{rep}$ on $X$ .

*Proof of Proposition 3.5.* On the Riemann surface  $X$ , we have  $F \circ \pi_X \circ g = Q \circ \varphi^{-1} \circ \pi_X \circ \tilde{\varphi} \circ \tilde{Q}^{-1} = Q \circ \tilde{Q}^{-1} = \pi_X$  (see Figure 4). At a small neighborhood of  $\infty$ ,  $F$  has an inverse branch  $\bar{g}(z) = z - (b_0 - c_0) + o(1)$  as  $z \rightarrow \infty$ . By Lemma 3.9 (a) and Lemma 3.28 (a), we have

$$|\arg(b_0 - c_0)| = \left| \arg\left(1 + \frac{c_{00} - c_0}{b_0 - c_{00}}\right) \right| \leq \arcsin\left(\frac{c_{01, \max}}{b_0 - c_{00}}\right) (\doteq 0.2900 \dots) < \frac{\pi}{10}. \quad (3.46^*)$$

Let  $L > 0$  be a large number. Then  $\bar{g}$  is defined and injective in  $W = \mathbb{C} \setminus \overline{\mathbb{V}}(-L, \frac{\pi}{10})$  and satisfies  $|\arg(\bar{g}(z) - z) - \pi| < \frac{\pi}{10}$ . Hence  $\bar{g}(W) \subset W$  and  $\text{Re } \bar{g}(z) < \text{Re } z - (b_0 - c_{00}) + c_{01, \max} + 1 < \text{Re } z - 4$ . The rest part of the proof can be obtained by a word for word copy of [IS08, §5.J].  $\square$

### 3.11. Attracting Fatou coordinate $\Phi_{attr}$ and domains $D_1$ , $D_1^\sharp$ and $D_1^\flat$ .

**Definition.** Let  $pr_+(z) = \text{Re}(ze^{-i\pi/5})$  and  $pr_-(z) = \text{Re}(ze^{+i\pi/5})$  be the orthogonal projection to the line with angle  $\pm \frac{\pi}{5}$  to the real axis respectively. Define

$$\begin{aligned} H_1^\pm &= \{z : pr_\pm(z) > u_1 := 7.8\}, & H_2^\pm &= \{z : pr_\pm(z) > u_2 := 5.4\}, \\ H_3^\pm &= \{z : pr_\pm(z) \geq u_3 := cv \cos(\frac{\pi}{5}) (\doteq 13.7677 \dots)\}, & H_4^\pm &= \{z : pr_\pm(z) \geq u_4 := 11.5\}. \end{aligned}$$

**Lemma 3.34** (Attracting Fatou coordinate  $\Phi_{attr}$ ). (a)  $\varphi(H_2^\pm) \supset H_1^\pm$  and  $\varphi(H_4^\pm) \supset H_3^\pm$ . Hence  $F$  is defined on  $H_1^+ \cup H_1^-$ .

(b)  $Q$  is injective in  $H_2^\pm$ . Therefore,  $F$  is injective in  $H_1^\pm$ .

(c) If  $z \in H_1^\pm$ , then  $|\arg(F(z) - z)| < \frac{3\pi}{10}$ , hence  $F(H_1^\pm) \subset H_1^\pm$ . In particular, the sector  $H_1^+ \cup H_1^- = \mathbb{V}(u_0, \frac{7\pi}{10})$  is forward invariant under  $F$  and contained in  $\text{Basin}(\infty)$ , where  $u_0 = u_1 / \cos(\frac{\pi}{5})$ .

(d) There exists an attracting Fatou coordinate  $\Phi_{attr}$  for  $F$  in  $\mathbb{V}(u_0, \frac{7\pi}{10})$  and it is injective in each of  $H_1^\pm$ .

In the following, we normalize the Fatou coordinate  $\Phi_{attr}$  such that  $\Phi_{attr}(cv) = 1$ .

*Proof.* (a) By Lemma 3.28 (b), the half planes  $H_1^\pm$  is contained in  $\text{Image}(\varphi)$ . If  $\zeta \in \partial H_2^\pm$ , then by Lemma 3.28 (e), we have

$$\begin{aligned} pr_\pm(\varphi(\zeta)) &= pr_\pm(\zeta) + pr_\pm(c_{00}) + pr_\pm(c_0 - c_{00}) + pr_\pm(\varphi_1(\zeta)) \\ &\leq 5.4 + c_{00} \cos(\frac{\pi}{5}) + c_{01, \max} + \varphi_{1, \max}(5.4) (\doteq 7.7399 \dots) < 7.8. \end{aligned} \quad (3.47^*)$$

Hence  $\varphi(\zeta) \notin H_1^\pm$ . This means that  $\varphi^{-1}(H_1^\pm)$  must be contained in one side of  $\partial H_2^\pm$ . On the other hand, if  $\zeta \in H_1^\pm$  is a point with large real part, then  $\varphi^{-1}(\zeta) \in H_2^\pm$ . This means that  $\varphi^{-1}(H_1^\pm) \subset H_2^\pm$  and hence  $\varphi(H_2^\pm) \supset H_1^\pm$ .

If  $\zeta \in \partial H_4^\pm$ , then

$$\begin{aligned} pr_\pm(\varphi(\zeta)) &= pr_\pm(\zeta) + pr_\pm(c_{00}) + pr_\pm(c_0 - c_{00}) + pr_\pm(\varphi_1(\zeta)) \\ &\leq 11.5 + c_{00} \cos(\frac{\pi}{5}) + c_{01, \max} + \varphi_{1, \max}(11.5) (\doteq 13.7266 \dots) \\ &< pr_\pm(cv) (\doteq 13.7677 \dots). \end{aligned} \quad (3.48^*)$$

The same argument as in (a) shows that  $\varphi(H_4^\pm) \supset H_3^\pm$ .

(b) By Lemma 3.27 (b),  $Q$  is a injection in  $H_2^\pm$ . Then  $F$  is injective in  $H_1^\pm$  since  $\varphi^{-1}(H_1^\pm) \subset H_2^\pm$  and  $F = Q \circ \varphi^{-1}$ .

(c) If  $z \in H_1^\pm$ , then  $\zeta = \varphi^{-1}(z) \in H_2^\pm$  by (a). By Lemma 3.32 (b), we have

$$\begin{aligned} |\arg(F(z) - z)| &\leq \max\{Arg\Delta F_{\max}(5.4, \pm\frac{\pi}{5}), -Arg\Delta F_{\min}(5.4, \pm\frac{\pi}{5})\} \\ &(\doteq \max\{0.4967 \dots, 0.7573 \dots\}) < \frac{3\pi}{10} (\doteq 0.9424 \dots). \end{aligned} \quad (3.49^*)$$

This implies that each of  $H_1^\pm$  is forward invariant under  $F$  and hence  $H_1^+ \cup H_1^- = \mathbb{V}(u_0, \frac{7\pi}{10})$  is also. The assertion that  $\mathbb{V}(u_0, \frac{7\pi}{10})$  is contained in  $\text{Basin}(\infty)$  follows immediately.

(d) can be proven as the proof of [IS08, Proposition 5.5].  $\square$

**Lemma 3.35** (Estimates on  $\Phi_{attr}$ ). (a) The attracting Fatou coordinate  $\Phi_{attr}$  satisfies the following inequalities:

$$-\frac{\pi}{5} < \arg \Phi'_{attr}(z) < \frac{\pi}{4} \text{ for } z \in H_3^+ \text{ and } -\frac{\pi}{4} < \arg \Phi'_{attr}(z) < \frac{\pi}{5} \text{ for } z \in H_3^-; \quad (3.50)$$

$$0.067 < |\Phi'_{attr}(z)| < 0.276 \text{ for } z \in H_3^+ \cup H_3^- = \overline{\mathbb{V}}(cv, \frac{7\pi}{10}). \quad (3.51)$$

(b)  $\Phi_{attr}$  is injective in  $H_3^+ \cup H_3^- = \overline{\mathbb{V}}(cv, \frac{7\pi}{10})$ . There exists a domain  $\mathcal{H}_1$  such that  $\Phi_{attr}$  is a homeomorphism from  $\overline{\mathcal{H}}_1$  onto  $\{z : \text{Re } z \geq 1\}$ , and  $\mathcal{H}_1$  satisfies  $\overline{\mathbb{V}}(cv, \frac{3\pi}{10}) \subset \mathcal{H}_1 \cup \{cv\} \subset \overline{\mathcal{H}}_1 \subset \mathbb{V}(cv, \frac{7\pi}{10}) \cup \{cv\}$  and  $cv \in \partial \mathcal{H}_1$ .

*Proof.* (a) Suppose  $z \in H_3^+$ . Then  $\zeta = \varphi^{-1}(z) \in H_4^+$ , i.e.  $\text{Re}(\zeta e^{-i\pi/5}) \geq u_4 = 11.5$ . We claim that

$$F(z) \in \mathbb{D}_{H_1^+}(z, s(r_4)) \quad (3.52)$$

with  $r_4 = 0.53$ , where  $s(r) = \log \frac{1+r}{1-r}$  is defined in Lemma 3.11 (b). By applying Lemma 3.11 (b) with  $H = H_1^+$ ,  $t = u_1$ ,  $u = pr_+(z) - u_1$ ,  $r = r_4$ ,  $\theta = \frac{\pi}{5}$ , (3.52) is equivalent to

$$F(z) - z \in \mathbb{D} \left( \frac{2ur_4^2 e^{i\pi/5}}{1-r_4^2}, \frac{2ur_4}{1-r_4^2} \right). \quad (3.53)$$

This disk contains 0 since  $|r_4^2 e^{i\pi/5}| < r_4$  and it is increasing with  $u$ . Therefore, we only need to check (3.53) with the smallest  $u$ , i.e.  $u_5 = u_3 - u_1 = pr_+(cv) - 7.8$ . As in the proof of [IS08, Lemma 5.27 (a)], one can write  $F(z) - z = \alpha + \beta$ , where  $\alpha = b_0 - c_{00} + \frac{b_1 e^{-i\pi/5}}{2u_4}$ ,  $|\beta| \leq \beta_{\max}(u_4)$  and  $u_4 = 11.5$ . By a numerical estimate, we have

$$\left| \alpha - \frac{2u_5 r_4^2 e^{i\pi/5}}{1-r_4^2} \right| + \beta_{\max}(u_4) - \frac{2u_5 r_4}{1-r_4^2} (\doteq -0.0947 \dots) < 0. \quad (3.54^*)$$

This means that (3.52) and (3.53) are true.

Applying Theorem 3.12 to  $\Phi_{attr}$  with  $\Omega = H_1^+$ ,  $r = r_4$  and by Lemma 3.32, we have

$$\begin{aligned} \arg \Phi'_{attr}(z) &\leq -\arg(F(z) - z) + \frac{1}{2} |\log F'(z)| + \frac{1}{2} \log \frac{1}{1-r_4^2} \\ &\leq -\text{Arg} \Delta F_{\min}(u_4, \frac{\pi}{5}) + \frac{1}{2} \text{Log} DF_{\max}(u_4) - \frac{1}{2} \log(1-r_4^2) \\ &(\doteq 0.7619 \dots) < \frac{\pi}{4} (\doteq 0.7853 \dots); \end{aligned} \quad (3.55^*)$$

$$\begin{aligned} \arg \Phi'_{attr}(z) &\geq -\arg(F(z) - z) - \frac{1}{2} |\log F'(z)| - \frac{1}{2} \log \frac{1}{1-r_4^2} \\ &\geq -\text{Arg} \Delta F_{\max}(u_4, \frac{\pi}{5}) - \frac{1}{2} \text{Log} DF_{\max}(u_4) + \frac{1}{2} \log(1-r_4^2) \\ &(\doteq -0.6260 \dots) > -\frac{\pi}{5} (\doteq -0.6283 \dots). \end{aligned} \quad (3.56^*)$$

The similar estimate can be applied to  $z \in H_3^-$ .

For  $|\Phi'_{attr}(z)|$  on  $H_3^+$  or  $H_3^-$ , by Theorem 3.12 and Lemma 3.32, we have

$$\begin{aligned} |\Phi'_{attr}(z)| &\leq \exp \left( -\log |F(z) - z| + \frac{1}{2} |\log F'(z)| + \frac{1}{2} \log \frac{1}{1-r_4^2} \right) \\ &\leq \frac{\exp(\frac{1}{2} \text{Log} DF_{\max}(u_4))}{\text{Abs} \Delta F_{\min}(u_4, \frac{\pi}{5}) \sqrt{1-r_4^2}} (\doteq 0.2756 \dots) < 0.276; \end{aligned} \quad (3.57^*)$$

$$\begin{aligned} |\Phi'_{attr}(z)| &\geq \exp \left( -\log |F(z) - z| - \frac{1}{2} |\log F'(z)| - \frac{1}{2} \log \frac{1}{1-r_4^2} \right) \\ &\geq \frac{\sqrt{1-r_4^2}}{\text{Abs} \Delta F_{\max}(u_4, \frac{\pi}{5}) \exp(\frac{1}{2} \text{Log} DF_{\max}(u_4))} (\doteq 0.0670 \dots) > 0.067. \end{aligned} \quad (3.58^*)$$

(b) Suppose that  $[z_1, z_2] \subset \overline{\mathbb{V}}(cv, \frac{7\pi}{10})$  is a non-trivial segment. Let  $\theta = -\frac{\pi}{4}$  and  $\theta' = \frac{\pi}{4}$ . Applying Lemma 3.13 with  $f = \Phi_{attr}$ , we have  $\text{Re} \frac{\Phi_{attr}(z_2) - \Phi_{attr}(z_1)}{z_2 - z_1} > 0$  and  $\Phi_{attr}(z_1) \neq \Phi_{attr}(z_2)$ . If  $z_1$  and  $z_2$  in  $\overline{\mathbb{V}}(cv, \frac{7\pi}{10})$  cannot be joined by a segment in  $\overline{\mathbb{V}}(cv, \frac{7\pi}{10})$ , then one can find a point  $z_3$  such that  $[z_1, z_3]$  and  $[z_3, z_2]$  are contained in  $\overline{\mathbb{V}}(cv, \frac{7\pi}{10})$  and  $\frac{3\pi}{10} \leq \arg(z_3 - z_1) \leq \frac{7\pi}{10}$  and  $\frac{3\pi}{10} \leq \arg(z_2 - z_3) \leq \frac{7\pi}{10}$  (interchanging  $z_1$  and  $z_2$  if necessary). By Lemma 3.13, we have  $0 < \frac{3\pi}{10} - \frac{\pi}{4} < \arg(\Phi_{attr}(z_3) - \Phi_{attr}(z_1)) < \frac{7\pi}{10} + \frac{\pi}{4} < \pi$ . The same estimate holds for  $z_2 - z_3$  and hence  $\text{Im}(\Phi_{attr}(z_2) - \Phi_{attr}(z_1)) > 0$ . Above all, it follows that  $\Phi_{attr}$  is injective in  $\overline{\mathbb{V}}(cv, \frac{7\pi}{10})$ .

Similarly, by Lemma 3.13, if  $z_1, z_2 \in H_3^+$  and  $z_1 \neq z_2$ , then

$$\arg(z_2 - z_1) - \frac{\pi}{5} < \arg(\Phi_{attr}(z_2) - \Phi_{attr}(z_1)) < \arg(z_2 - z_1) + \frac{\pi}{4}. \quad (3.59)$$

In particular, if  $\arg(z - cv) = \frac{7\pi}{10}$  (now  $z$  is on the boundary of  $\bar{\mathbb{V}}(cv, \frac{7\pi}{10})$ ), note that  $\Phi_{attr}(cv) = 1$ , we have

$$\frac{\pi}{2} = \frac{7\pi}{10} - \frac{\pi}{5} < \arg(\Phi_{attr}(z) - 1) < \frac{7\pi}{10} + \frac{\pi}{4} < \pi.$$

This means that  $\operatorname{Re}(\Phi_{attr}(z) - 1) < 0$  and  $\operatorname{Im}(\Phi_{attr}(z) - 1) > 0$ . A similar estimate holds for  $H_3^-$ . Also, by (3.3), we have

$$\left| \frac{\Phi_{attr}(z) - 1}{z - cv} \right| \geq \int_0^1 \operatorname{Re} \Phi'_{attr}(cv + t(z - cv)) dt \geq 0.067 \cos(\frac{\pi}{4}) = 0.067/\sqrt{2} > 0.$$

Therefore, as  $z \rightarrow \infty$  in  $\bar{\mathbb{V}}(cv, \frac{7\pi}{10})$ , then  $\Phi_{attr}(z) \rightarrow \infty$ .

Given  $R' > 0$ , take  $R'' > \sqrt{2}R'/0.067$  and denote  $G = \mathbb{V}(cv, \frac{7\pi}{10}) \cap \mathbb{D}(cv, R'')$ . The results above imply that  $\Phi_{attr}(\partial G)$  is disjoint with  $\{z : \operatorname{Re} z \geq 1\} \cap \bar{\mathbb{D}}(1, R')$  except at 1. By Argument Principle, one can prove that  $\{z : \operatorname{Re} z \geq 1\} \cap \bar{\mathbb{D}}(1, R') \subset \Phi_{attr}(G) \cup \{1\}$ . Since  $R'$  is arbitrary,  $\{z : \operatorname{Re} z \geq 1\}$  is contained in the image of  $\mathbb{V}(cv, \frac{7\pi}{10}) \cup \{cv\}$  by  $\Phi_{attr}$ . Define

$$\mathcal{H}_1 = \Phi_{attr}^{-1}(\{z : \operatorname{Re} z > 1\}).$$

If  $z \in \bar{\mathbb{V}}(cv, \frac{3\pi}{10}) = H_3^+ \cap H_3^-$ , by (3.59), where  $\frac{\pi}{4}$  can be replaced by  $\frac{\pi}{5}$ , we have  $|\arg(\Phi_{attr}(z) - 1)| < \frac{3\pi}{10} + \frac{\pi}{5} = \frac{\pi}{2}$ . Therefore,  $\Phi_{attr}(\bar{\mathbb{V}}(cv, \frac{3\pi}{10}))$  is contained in  $\{z : \operatorname{Re} z > 1\} \cup \{1\}$ . Hence we have  $\bar{\mathbb{V}}(cv, \frac{3\pi}{10}) \subset \mathcal{H}_1 \cup \{cv\} \subset \bar{\mathcal{H}}_1 \subset \mathbb{V}(cv, \frac{7\pi}{10}) \cup \{cv\}$ .  $\square$

*Proof of Proposition 3.6.* We only need to prove (b) since (a) is covered by Lemmas 3.34 and 3.35. Define  $D_1 = \Phi_{attr}^{-1}(\{z : 1 < \operatorname{Re} z < 2, -\eta < \operatorname{Im} z < \eta\})$ ,  $D_1^\sharp = \Phi_{attr}^{-1}(\{z : 1 < \operatorname{Re} z < 2, \operatorname{Im} z > \eta\})$  and  $D_1^\flat = \Phi_{attr}^{-1}(\{z : 1 < \operatorname{Re} z < 2, \operatorname{Im} z < -\eta\})$ , where the inverse is taken only in  $\bar{\mathbb{V}}(cv, \frac{7\pi}{10})$ . If  $|\arg(z - F(cv))| \leq \frac{3\pi}{10}$ , then  $z \in \bar{\mathbb{V}}(cv, \frac{3\pi}{10})$  since  $F(cv) \in \bar{\mathbb{V}}(cv, \frac{\pi}{5})$  by Lemma 3.33. As before, by (3.59), where  $\frac{\pi}{4}$  can be replaced by  $\frac{\pi}{5}$ , we have  $|\arg(\Phi_{attr}(z) - \Phi_{attr}(F(cv)))| < \frac{\pi}{2}$ . Therefore,  $\operatorname{Re} \Phi_{attr}(z) > \operatorname{Re} \Phi_{attr}(F(cv)) = 2$ . This means that  $D_1$ ,  $D_1^\sharp$  and  $D_1^\flat$  are all contained in  $W_1$ . Similarly, if  $|\arg(z - cv)| \leq \frac{\pi}{5}$ , then  $|\arg(\Phi_{attr}(z) - 1)| < \frac{\pi}{5} + \frac{\pi}{5} = \frac{2\pi}{5}$ . Then  $\Phi_{attr}(z)$  cannot be in  $\{z : 1 < \operatorname{Re} z < 2, |\operatorname{Im} z| > \eta\}$  since<sup>13</sup>

$$\tan \frac{2\pi}{5} = \sqrt{5 + 2\sqrt{5}} (\doteq 3.0776 \dots) < \eta = 3.1. \quad (3.60^*)$$

This means that  $D_1^\sharp$  and  $D_1^\flat$  are contained in  $\{z : \frac{\pi}{5} \leq \pm \arg(z - cv) < \frac{7\pi}{10}\}$ . Finally, it remains to show  $D_1 \subset \mathbb{D}(cv, R_1)$ . Note that the derivative of  $\Phi_{attr}^{-1}$  is bounded by  $\frac{1}{0.067}$  and  $\{z : 1 < \operatorname{Re} z < 2, \eta < \operatorname{Im} z < \eta\} \subset \mathbb{D}(1, \sqrt{1 + \eta^2})$ , we have  $D_1 \subset \mathbb{D}(cv, \sqrt{1 + \eta^2}/0.067)$ . Therefore, we only need to check that  $\sqrt{1 + \eta^2}/0.067 < R_1 = 82$ . Actually, this is true even for a bigger  $\eta$  such as  $\eta = 5.4$  since

$$\sqrt{1 + 5.4^2}/0.067 (\doteq 81.9673 \dots) < 82. \quad (3.61^*)$$

The proof of Proposition 3.6 is complete.  $\square$

<sup>13</sup>Maybe the lower bound of  $\eta$  can be improved such that it is less than 3. But this is not important here.

**3.12. Locating domains**  $D_0$ ,  $D'_0$ ,  $D''_0$ ,  $D_{-1}$ ,  $D'''_{-1}$  and  $D''''_{-1}$ . Let  $\Omega_1$ ,  $\Omega_2$  and  $\Omega_3$  are unbounded sets defined in Lemma 3.30. For  $i = 1, 2, 3$ , we denote  $\tilde{\Omega}_i = \zeta(\Omega_i)$  and  $\tilde{\Omega}_i^\pm := \tilde{\Omega}_i \cap \{\zeta : \pm \operatorname{Im} \zeta > 0\}$ .

**Lemma 3.36.** (a) Let  $\tilde{W}_0 := \{\zeta : \operatorname{Re} \zeta > cp + 0.5 \text{ or } pr_+(\zeta) > 7.5 \text{ or } pr_-(\zeta) > 7.5\} \cup \{\zeta : pr_+(\zeta) > cp \cos(\frac{\pi}{5}) \text{ and } pr_-(\zeta) > cp \cos(\frac{\pi}{5})\}$ . Then  $\mathbb{V}(cv, \frac{7\pi}{10}) \subset Q(\tilde{W}_0) \subset \mathbb{C} \setminus (-\infty, cv]$  and  $\tilde{W}_0 \subset \mathcal{U}_1$ .

(b)  $\varphi(\tilde{W}_0) \subset W_0 := \{z : \operatorname{Re} z > 2 \text{ or } pr_+(z) > 5.3 \text{ or } pr_-(z) > 5.3\}$ .

(c)  $(Q^{-1}(W_0) \cap \mathcal{U}_{123}) \setminus (\overline{\mathbb{D}}(a_6, \varepsilon_6) \cup \overline{\mathbb{D}}(\bar{a}_6, \varepsilon_6)) \subset \tilde{W}_{-1} := (\tilde{\Omega}_0 \cup \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \tilde{\Omega}_3) \setminus (\overline{\mathbb{D}}(a_6, \varepsilon_6) \cup \overline{\mathbb{D}}(\bar{a}_6, \varepsilon_6))$ , where  $\tilde{\Omega}_0 := \mathbb{V}(0, 0.72\pi) \cap \{\zeta : \operatorname{Re} \zeta \leq x_E \text{ and } |\operatorname{Im} \zeta| > 2.7\}$ .

(d) If  $\zeta \in Q^{-1}(W_0) \cap (\bar{\mathcal{U}}_{2+} \cup \bar{\mathcal{U}}_{3-})$ , then  $\zeta \in (\bar{\mathcal{U}}_{2+} \cup \bar{\mathcal{U}}_{3-}) \setminus (\overline{\mathbb{D}}(a_5, \varepsilon_5) \cup \overline{\mathbb{D}}(\bar{a}_5, \varepsilon_5))$ .

See Figure 12. This is the most delicate lemma among all the estimations in this paper. We postpone the proof of Lemma 3.36 until the end of this subsection.

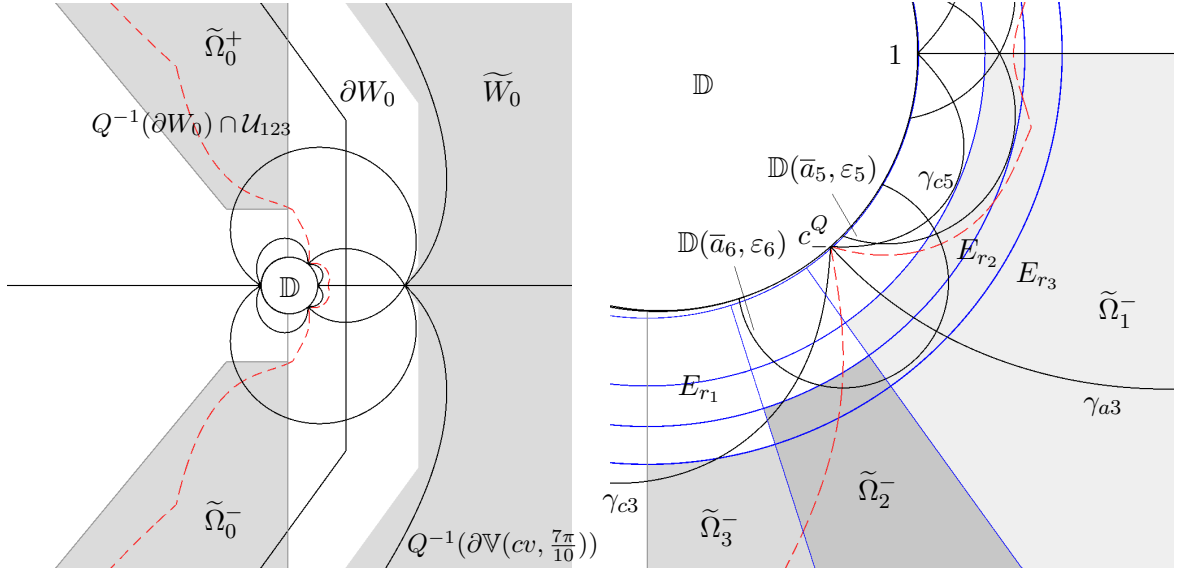


Figure 12: Left: The domains  $\tilde{\Omega}_0^+$ ,  $\tilde{\Omega}_0^-$ ,  $\tilde{W}_0$ ,  $W_0$  and their boundaries; Right: The zoom of the left picture near the unit circle. The sets  $\tilde{\Omega}_1^-$ ,  $\tilde{\Omega}_2^-$  and  $\tilde{\Omega}_3^-$  and some disks are shown. The curve  $Q^{-1}(W_0) \cap \mathcal{U}_{123}$  is depicted by a dashed line.

**Definition.** Note that  $Q$  maps  $\mathcal{U}_1$ ,  $\mathcal{U}_2$  and  $\mathcal{U}_3$  homeomorphically onto  $\mathbb{C} \setminus (-\infty, cv]$ . Define

$$\begin{aligned} \tilde{\mathcal{H}}_0 &= (Q|_{\mathcal{U}_1})^{-1}(\mathcal{H}_1), \quad \tilde{D}_0 = (Q|_{\mathcal{U}_1})^{-1}(D_1), \quad \tilde{D}_0^\sharp = (Q|_{\mathcal{U}_1})^{-1}(D_1^\sharp), \\ \tilde{D}'_0 &= (Q|_{\mathcal{U}_2})^{-1}(D_1) \text{ and } \tilde{D}''_0 = (Q|_{\mathcal{U}_3})^{-1}(D_1). \end{aligned}$$

The domains  $\tilde{\mathcal{H}}_0$  and hence  $\tilde{D}_0$  and  $\tilde{D}_0^\sharp$  are contained in  $\mathbb{C} \setminus E_{r_1}$  because of Lemma 3.24 (d) and  $\mathcal{H}_1 \subset \overline{\mathbb{V}}(cv, \frac{7\pi}{10}) \subset \mathbb{C} \setminus \overline{\mathbb{D}}(0, \rho)$ . The domains  $\tilde{D}'_0$  and  $\tilde{D}''_0$  are contained in  $\mathbb{C} \setminus E_{r_1}$  by Lemma 3.37 (b) below. Hence we can define

$$\mathcal{H}_0 = \varphi(\tilde{\mathcal{H}}_0), \quad D_0 = \varphi(\tilde{D}_0), \quad D_0^\sharp = \varphi(\tilde{D}_0^\sharp), \quad D'_0 = \varphi(\tilde{D}'_0), \quad D''_0 = \varphi(\tilde{D}''_0).$$

It is easy to see that  $F(\mathcal{H}_0) = \mathcal{H}_1$  and  $\Phi_{attr}$  extends naturally to  $\overline{\mathcal{H}}_0$  such that it is a homeomorphism onto  $\{z : \operatorname{Re} z \geq 0\}$ . Moreover,  $\Phi_{attr}(D_0) = \{z : 0 < \operatorname{Re} z < 1, |\operatorname{Im} z| < \eta\}$  and  $D_0 \subset \mathcal{H}_0 \setminus \overline{\mathcal{H}}_1$ . In particular,  $D_0 \cap \overline{\mathbb{V}}(cv, \frac{3\pi}{10}) = \emptyset$  since  $\overline{\mathbb{V}}(cv, \frac{3\pi}{10}) \subset \overline{\mathcal{H}}_1$ . By Lemma

3.36 (a) and (b),  $D_0$  is contained in  $W_0$  since  $D_1 \subset \mathbb{V}(cv, \frac{7\pi}{10})$ . This means that  $D_0$  must be contained in  $\mathbb{C} \setminus ((-\infty, 0] \cup [cv, +\infty))$ .

Note that  $Q$  maps  $(\mathcal{U}_{1+} \cup \mathcal{U}_{2-} \cup \gamma_{b1})$ ,  $(\mathcal{U}_{2+} \cup \mathcal{U}_{3-} \cup \gamma_{b2})$  and  $(\mathcal{U}_{1-} \cup \mathcal{U}_{3+} \cup \gamma_{b3})$  homeomorphically onto the double slitted complex plane  $\mathbb{C} \setminus ((-\infty, 0] \cup [cv, +\infty))$ . Thus we can define

$$\begin{aligned} \tilde{D}_{-1} &= (Q|_{(\mathcal{U}_{1+} \cup \mathcal{U}_{2-} \cup \gamma_{b1})})^{-1}(D_0), \quad \tilde{D}_{-1}''' = (Q|_{(\mathcal{U}_{2+} \cup \mathcal{U}_{3-} \cup \gamma_{b2})})^{-1}(D_0), \\ \text{and } \tilde{D}_{-1}'''' &= (Q|_{(\mathcal{U}_{1-} \cup \mathcal{U}_{3+} \cup \gamma_{b3})})^{-1}(D_0). \end{aligned}$$

These domains are contained in  $\mathbb{C} \setminus E_{r_1}$  by Lemma 3.37 (b) below. Then one can define

$$D_{-1} = \varphi(\tilde{D}_{-1}), \quad D_{-1}' = \varphi(\tilde{D}_{-1}'), \quad \text{and } D_{-1}'' = \varphi(\tilde{D}_{-1}'').$$

By the construction, we know that  $F$  maps  $D_0$ ,  $D_0'$ ,  $D_0''$ ,  $D_{-1}$ ,  $D_{-1}'$  and  $D_{-1}''$  homeomorphically onto  $D_1$ ,  $D_1$ ,  $D_1$ ,  $D_0$ ,  $D_0$  and  $D_0$  respectively.

Recall that  $R = 100$  and  $R_1 = 82$  are constants defined in §3.1.

**Lemma 3.37.** (a)  $\tilde{D}_0 \subset \tilde{W}_0 \cap \mathbb{D}(10, R_1 + 2)$ ;  $D_0 \subset W_0 \cap \mathbb{D}(10, R_1 + 5)$ .  
 (b)  $\tilde{D}_0 \cup \tilde{D}_0' \cup \tilde{D}_0'' \cup \tilde{D}_{-1} \cup \tilde{D}_{-1}' \cup \tilde{D}_{-1}'' \subset \tilde{W}_{-1} \cap \mathbb{D}(0, R_1 + 12) \cap \mathcal{U}_{123} \cap (\mathbb{C} \setminus E_{r_1})$ .  
 (c)  $D_0 \cup D_0' \cup D_0'' \cup D_{-1} \cup D_{-1}' \cup D_{-1}'' \subset \mathbb{D}(0, R_1 + 15)$ .

*Proof.* (a) If  $|\zeta| \geq 50$ , then

$$|Q(\zeta) - (\zeta + b_0)| \leq \frac{b_1}{50} + Q_{2, \max}(50) (\div 0.4382 \dots) < 1. \quad (3.62^*)$$

Hence if  $|\zeta - 10| \geq R_1 + 2$ , then  $|\zeta| > 50$  and  $|\zeta - (cv - b_0)| \geq |\zeta - 10| - |10 + b_0 - cv| > R_1 + 1$  since  $|10 + b_0 - cv| < 1$ . Therefore, we have  $|Q(\zeta) - cv| = |(Q(\zeta) - (\zeta + b_0)) + (\zeta - (cv - b_0))| \geq |\zeta - (cv - b_0)| - |Q(\zeta) - (\zeta + b_0)| > R_1$ . By Proposition 3.6 (b),  $D_1 \subset \mathbb{D}(cv, R_1)$ . This means that

$$\tilde{D}_0 \cup \tilde{D}_0' \cup \tilde{D}_0'' \subset Q^{-1}(D_1) \subset Q^{-1}(\mathbb{D}(cv, R_1)) \subset \mathbb{D}(10, R_1 + 2).$$

If  $\zeta \in \mathbb{C} \setminus E$  and  $|\zeta - 10| < R_1 + 2$ , then the Jordan curve  $\varphi(\{\zeta' : |\zeta' - 10| = R_1 + 2\})$  is contained in  $\mathbb{D}(10, R_1 + 5)$  by (3.43\*). This means that  $D_0 \cup D_0' \cup D_0'' \subset \mathbb{D}(10, R_1 + 5)$ . By Lemma 3.36 (a) and (b), we have  $\tilde{D}_0 \subset \tilde{W}_0$  and  $D_0 \subset W_0$ .

(b) By a similar argument, we have  $\tilde{D}_{-1} \cup \tilde{D}_{-1}' \cup \tilde{D}_{-1}'' \subset Q^{-1}(D_0) \subset Q^{-1}(\mathbb{D}(10, R_1 + 5)) \subset \mathbb{D}(3, R_1 + 7)$  and  $D_{-1} \cup D_{-1}' \cup D_{-1}'' \subset \mathbb{D}(3, R_1 + 10)$ . Let  $\zeta \in \tilde{D}_0 \cup \tilde{D}_0' \cup \tilde{D}_0'' \cup \tilde{D}_{-1} \cup \tilde{D}_{-1}' \cup \tilde{D}_{-1}''$ . By the above, we have  $\zeta \in \mathbb{D}(10, R_1 + 2) \cup \mathbb{D}(3, R_1 + 7) \subset \mathbb{D}(0, R_1 + 12)$ .

By Proposition 3.6 (b),  $D_1 \subset W_1 \subset \mathbb{V}(cv, \frac{7\pi}{10}) \subset W_0$ . Hence we have  $\zeta \in Q^{-1}(D_0 \cup D_1) \subset Q^{-1}(W_0)$ . By definition,  $\zeta \in \mathcal{U}_{123}$ . Note that

$$D_0 \cup D_1 \subset W_0 \cap (\mathbb{D}(10, R_1 + 5) \cup \mathbb{D}(cv, R_1)) \subset \overline{\mathbb{D}}(0, R) \setminus \overline{\mathbb{D}}(0, \rho), \quad (3.63)$$

i.e.  $\rho < |Q(\zeta)| < R$ . This means that  $\zeta \notin \overline{\mathbb{D}}(a_6, \varepsilon_6) \cup \overline{\mathbb{D}}(\bar{a}_6, \varepsilon_6)$  by Lemma 3.24 (a). We now divide the arguments into two cases. The first case, if  $\zeta \in Q^{-1}(W_0) \cap \mathcal{U}_{12}'$ , it follows from Lemma 3.24 (d) that  $\zeta \in \mathbb{C} \setminus E_{r_1}$ . The second case, if  $\zeta \in Q^{-1}(W_0) \cap (\overline{\mathcal{U}}_{2+} \cup \overline{\mathcal{U}}_{3-})$ , then  $\zeta \in (\overline{\mathcal{U}}_{2+} \cup \overline{\mathcal{U}}_{3-}) \setminus (\overline{\mathbb{D}}(a_5, \varepsilon_5) \cup \overline{\mathbb{D}}(\bar{a}_5, \varepsilon_5))$  by Lemma 3.36 (d). This concludes that  $\zeta \in \mathbb{C} \setminus E_{r_1}$  by Lemma 3.24 (c). Therefore, in either case,  $\zeta$  is contained in  $\mathbb{C} \setminus E_{r_1}$ . By Lemma 3.36 (c),  $\zeta$  is contained in  $\tilde{W}_{-1}$ .

(c) By (3.43\*), the Jordan curve  $\varphi(\{\zeta' : |\zeta'| = R_1 + 12\})$  is contained in  $\mathbb{D}(0, R_1 + 15)$ . Therefore, by (b), we have  $D_0 \cup D_0' \cup D_0'' \cup D_{-1} \cup D_{-1}' \cup D_{-1}'' \subset \mathbb{D}(0, R_1 + 15)$ .  $\square$

*Proof of Proposition 3.7.* We only need to prove (e). Lemma 3.37 (c) shows that  $\overline{D}_0 \cup \overline{D}_0' \cup \overline{D}_0'' \cup \overline{D}_{-1} \cup \overline{D}_{-1}' \cup \overline{D}_{-1}'' \setminus \{cv\} \subset \mathbb{D}(0, R)$  since  $R_1 + 15 = 97 < R = 100$ .

Let  $\zeta \in \text{closure}(\tilde{D}_0 \cup \tilde{D}_0' \cup \tilde{D}_0'' \cup \tilde{D}_{-1} \cup \tilde{D}_{-1}' \cup \tilde{D}_{-1}'') \subset \overline{\mathcal{U}}_{123}$ . Lemma 3.37 (b) implies that  $\zeta \in \mathbb{C} \setminus \text{int } E_{r_1}$ . By Lemma 3.29, we have  $|\varphi(\zeta)| > \rho$ . By Lemma 3.37 (b), we have  $\zeta \in \tilde{W}_{-1}$ .

By Lemma 3.36 (c), we know that  $\zeta \in \zeta(\Omega_1 \cup \Omega_2 \cup \Omega_3)$  or  $\zeta \in \widetilde{\Omega}_0$ . If  $\zeta \in \zeta(\Omega_1 \cup \Omega_2 \cup \Omega_3)$ , then  $\varphi(\zeta) \notin \mathbb{R}_-$  by Lemma 3.30. If  $\zeta \in \Omega_0$ , then  $|\operatorname{Im} \zeta| > 2.7$ . However, if  $|\zeta| > 2.7$ , similar to the argument of (3.43\*), we have

$$|\varphi(\zeta) - \zeta| \leq c_{00} + c_{01, \max} + \varphi_{1, \max}(2.7) (\doteq 2.5795 \dots) < 2.7. \quad (3.64^*)$$

Therefore,  $\varphi(\zeta) \notin \mathbb{R}_-$  in this case.

Let  $z \in \overline{D}_0 \cup \overline{D}'_0 \cup \overline{D}''_0 \cup \overline{D}_{-1} \cup \overline{D}'''_{-1} \cup \overline{D}''''_{-1}$ . Then  $F(z) \in \overline{\mathcal{H}}_0$  and  $0 \leq \operatorname{Re} \Phi_{\text{attr}}(F(z)) \leq 2$ . By Lemma 3.35 (b), if  $z' \in \overline{\mathbb{V}}(cv, \frac{\pi}{5})$  but  $z' \neq cv$ , we have  $\operatorname{Re} \Phi_{\text{attr}}(z') > 1$  and hence  $\operatorname{Re} \Phi_{\text{attr}}(F(z')) > 2$ . Therefore,  $z$  cannot lie in  $\overline{\mathbb{V}}(cv, \frac{\pi}{5}) \setminus \{cv\}$ . This ends the proof of the first statement in Proposition 3.7 (e).

Finally, we need to prove that  $\overline{D}_0^\sharp \subset \pi_X(X_{1+})$ . Similar to the argument in last paragraph, one can prove that if  $z \in \overline{D}_0^\sharp$ , then  $z \notin \overline{\mathbb{V}}(cv, \frac{\pi}{5})$ . By Proposition 3.6 (b) and the definition of  $W_1$ , we have  $D_1^\sharp \subset W_1 \subset \mathbb{V}(cv, \frac{7\pi}{10})$ . By Lemma 3.36 (a) and (b), we have  $\widetilde{D}_0^\sharp \subset \widetilde{W}_0$  and  $\overline{D}_0^\sharp \subset \overline{W}_0$ . Therefore,  $\overline{D}_0^\sharp \cap \overline{\mathbb{D}}(0, \rho) = \emptyset$ . This means that  $\overline{D}_0^\sharp \subset \pi_X(X_{1+})$ .  $\square$

The rest of this subsection will devote to the proof of Lemma 3.36.

*Proof of Lemma 3.36.* (a) The boundary  $\partial \widetilde{W}_0$  consists of  $\ell_0^\pm : \zeta = cp + te^{\pm \frac{3\pi i}{10}} (0 \leq t \leq t_0)$ ,  $\ell_1^\pm : \zeta = cp + 0.5 \pm it (t_1 \leq t \leq t_2)$  and  $\ell_2^\pm : \zeta = cp + 0.5 \pm it_2 + te^{\pm \frac{7\pi i}{10}} (t \geq 0)$ , where

$$t_0 = \frac{0.5}{\sin(\frac{\pi}{5})}, \quad t_1 = \frac{0.5}{\tan(\frac{\pi}{5})} \quad \text{and} \quad t_2 = \tan(\frac{3\pi}{10}) \left( \frac{7.5}{\cos(\frac{\pi}{5})} - cp - 0.5 \right).$$

We first prove that  $Q(\ell_0^\pm), Q(\ell_1^\pm), Q(\ell_2^\pm) \subset \{z : \frac{7\pi}{10} < \pm \arg(z - cv) < \pi\} \cup \{cv\}$ . Note that  $cp$  is the critical point of  $Q$  with local degree 3, hence  $Q$  can be written as  $Q(\zeta) - cv = \alpha_3(\zeta - cp)^3(1 + Q_1(\zeta))$  near  $cp$ , where  $Q_1(\zeta) = \mathcal{O}(\zeta - cp)$  if  $\zeta$  is close to  $cp$ . The constant  $\alpha_3$  can be calculated by  $\alpha_3 = Q'''(cp)/6 \in \mathbb{R}$ . For  $Q_1(\zeta)$ , we have the estimation

$$\max_{\zeta: |\zeta - cp| = 0.4} |Q_1(\zeta)| = \max_{\theta \in [0, 2\pi]} \left| \frac{Q(cp + 0.4e^{i\theta}) - cv}{\alpha_3(0.4e^{i\theta})^3} - 1 \right| (\doteq 0.2885 \dots) < 0.29. \quad (3.65^*)$$

By Lemma 3.9, if  $\zeta = cp + te^{\frac{3\pi i}{10}}$  for  $0 \leq t \leq 0.4$ , we have

$$\begin{aligned} \frac{7\pi}{10} &< 3 \times \frac{3\pi}{10} - \frac{\pi}{3} \times 0.29 < \arg(Q(\zeta) - cv) = 3\arg(\zeta - cp) + \arg(1 + Q_1(\zeta)) \\ &< 3 \times \frac{3\pi}{10} + \frac{\pi}{3} \times 0.29 < \pi. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \max_{t \in [0.4, t_0]} \arg(Q(cp + te^{\frac{3\pi i}{10}}) - cv) &(\doteq 2.6443 \dots) < \pi, \quad \text{and} \\ \min_{t \in [0.4, t_0]} \arg(Q(cp + te^{\frac{3\pi i}{10}}) - cv) &(\doteq 2.4641 \dots) > \frac{7\pi}{10} (\doteq 2.1991 \dots). \end{aligned} \quad (3.66^*)$$

Therefore, we have  $Q(\ell_0^+) \subset \{z : \frac{7\pi}{10} < \arg(z - cv) < \pi\} \cup \{cv\}$ .

Next, we consider the maximum and minimum of  $\arg(Q(\zeta) - cv)$  when  $\zeta = cp + 0.5 + it (t_1 \leq t \leq t_2) \in \ell_1^+$ . The numerical estimation shows that

$$\begin{aligned} \max_{t \in [t_1, t_2]} \arg(Q(cp + 0.5 + it) - cv) &(\doteq 2.9909 \dots) < \pi, \quad \text{and} \\ \min_{t \in [t_1, t_2]} \arg(Q(cp + 0.5 + it) - cv) &(\doteq 2.3140 \dots) > \frac{7\pi}{10} (\doteq 2.1991 \dots). \end{aligned} \quad (3.67^*)$$

Therefore, we have  $Q(\ell_1^+) \subset \{z : \frac{7\pi}{10} < \arg(z - cv) < \pi\}$ .

Finally, we consider  $\zeta = cp + 0.5 + it_2 + t e^{\frac{7\pi i}{10}} (t \geq 0) \in \ell_2^+$ . If  $t \geq 100$ , then

$$\begin{aligned} |Q(\zeta) - \zeta - b_0| &< \max_{|\zeta|=100} |Q(\zeta) - \zeta - b_0| (\doteq 0.2166\dots) \\ &< (cv - \frac{7.5}{\cos(\frac{\pi}{5})} - b_0) \cos(\frac{\pi}{5}) (\doteq 0.3233\dots). \end{aligned} \quad (3.68^*)$$

This means that the image of  $\zeta = cp + 0.5 + it_2 + t e^{\frac{7\pi i}{10}} (t \geq 100)$  under  $Q$  is contained in  $\{z : \frac{7\pi}{10} < \arg(z - cv) < \pi\}$ . For the part of  $0 \leq t \leq 100$ , we have

$$\begin{aligned} \max_{t \in [0, 100]} \arg(Q(cp + 0.5 + it_2 + t e^{\frac{7\pi i}{10}}) - cv) &(\doteq 2.3504\dots) < \pi, \text{ and} \\ \min_{t \in [0, 100]} \arg(Q(cp + 0.5 + it_2 + t e^{\frac{7\pi i}{10}}) - cv) &(\doteq 2.2038\dots) > \frac{7\pi}{10} (\doteq 2.1991\dots). \end{aligned} \quad (3.69^*)$$

Therefore, we have  $Q(\ell_2^+) \subset \{z : \frac{7\pi}{10} < \arg(z - cv) < \pi\}$ . By the symmetry of the dynamics of  $Q$ , the same estimation holds for  $Q(\ell_0^-)$ ,  $Q(\ell_1^-)$  and  $(\ell_2^-)$ .

The assertion  $\mathbb{V}(cv, \frac{7\pi}{10}) \subset Q(\widetilde{W}_0)$  can be proved similar as in Lemma 3.35 (b). Since  $Q(\partial\widetilde{W}_0)$  is disjoint with  $\Gamma_b^Q = (0, cv]$  except at  $cv$ , hence  $\partial\widetilde{W}_0$  is disjoint with  $\gamma_{b1} \cup \gamma_{b3} \cup \{-1\}$  except at  $cp$ . Since  $\partial\widetilde{W}_0$  is unbounded, it must be contained in the unbounded component of  $\mathbb{C} \setminus (\gamma_{b1} \cup \gamma_{b3} \cup \{-1\})$ , which is  $\mathcal{U}_1 \cup (-\infty, -1)$ . But obviously,  $\widetilde{W}_0 \cap (-\infty, -1) = \emptyset$ . Hence  $\widetilde{W}_0 \subset \mathcal{U}_1$  and  $Q(\widetilde{W}_0) \subset \mathbb{C} \setminus (-\infty, cv]$ .

(b) Suppose  $\zeta \in \widetilde{W}_0$ . Then there are three cases: (i)  $pr_{\pm}(\zeta) \geq 7.5$ ; (ii)  $\operatorname{Re} \zeta \geq cp + 0.5$  and (iii)  $pr_+(\zeta) > cp \cos(\frac{\pi}{5})$  and  $pr_-(\zeta) > cp \cos(\frac{\pi}{5})$ . For case (i), if  $pr_{\pm}(\zeta) \geq 7.5$ , then

$$\begin{aligned} pr_{\pm}(\varphi(\zeta)) &= pr_{\pm}(\zeta) + pr_{\pm}(c_{00}) + pr_{\pm}(c_0 - c_{00}) + pr_{\pm}(\varphi_1(\zeta)) \\ &\geq 7.5 + c_{00} \cos(\frac{\pi}{5}) - c_{01, \max} - \varphi_{1, \max}(7.5) (\doteq 5.3060\dots) > 5.3. \end{aligned} \quad (3.70^*)$$

For case (ii), if  $\operatorname{Re} \zeta \geq cp + 0.5$ , then

$$\operatorname{Re} \varphi(\zeta) \geq (cp + 0.5) + c_{00} - c_{01, \max} - \varphi_{1, \max}(cp + 0.5) (\doteq 2.3013\dots) > 2. \quad (3.71^*)$$

For case (iii), we only need to consider the restriction of  $\varphi$  on the segment  $[cp, (cp + 0.5) + 0.5 \tan(0.3\pi)i]$ . This segment can be written as

$$\zeta_t = cp + 0.5t(1 + \tan(0.3\pi)i), \text{ where } t \in [0, 1].$$

Recall that  $\zeta = \zeta(w) = e_1 w + e_0 + \frac{e-1}{w}$  is a conformal mapping from  $\mathbb{C} \setminus \overline{\mathbb{D}}$  onto  $\mathbb{C} \setminus E$ . Let  $w_t \in \mathbb{C} \setminus \overline{\mathbb{D}}$  be the unique point such that  $e_1 w_t + e_0 + \frac{e-1}{w_t} = \zeta_t$ . Then

$$w_t = \frac{\zeta_t - e_0 + \sqrt{(\zeta_t - e_0)^2 - 4e_1 e_{-1}}}{2e_1}.$$

By (3.42) and Lemma 3.28 (c), we have

$$|\arg \varphi(\zeta(w_t))| \leq |\arg w_t| + \log \frac{|w_t|+1}{|w_t|-1} \text{ and } |\varphi(\zeta(w_t))| \geq e_1 |w_t| \left(1 - \frac{1}{|w_t|}\right)^2.$$

Combine the results above, we have

$$\begin{aligned} \operatorname{Re} \varphi(\zeta_t) &= \operatorname{Re} \varphi(\zeta(w_t)) = |\varphi(\zeta(w_t))| \cos(\arg \varphi(\zeta(w_t))) \\ &\geq \min_{t \in [0, 1]} e_1 |w_t| \left(1 - \frac{1}{|w_t|}\right)^2 \cos \left( |\arg w_t| + \log \frac{|w_t|+1}{|w_t|-1} \right) (\doteq 2.0104\dots) > 2. \end{aligned} \quad (3.72^*)$$

Therefore, in either case, we have  $\varphi(\zeta) \in W_0$ .

(c) It is sufficient to prove that  $Q^{-1}(W_0) \cap \mathcal{U}_{123} \subset \widetilde{\Omega}_0 \cup \widetilde{\Omega}_1 \cup \widetilde{\Omega}_2 \cup \widetilde{\Omega}_3 \cup \mathbb{D}(a_6, \varepsilon_6) \cup \mathbb{D}(\bar{a}_6, \varepsilon_6)$ . Denote  $\widetilde{\Omega}_0^{\pm} := \widetilde{\Omega}_0 \cap \{\zeta : \pm \operatorname{Im} \zeta > 0\}$ . By the symmetry, we only consider the estimation in the lower half plane. Recall that  $r_1 = 1.25$ ,  $r_2 = 1.4$  and  $r_3 = 1.54$  are defined in Lemma 3.30. The boundary of  $\widetilde{\Omega}_0^- \cup \widetilde{\Omega}_1^- \cup \widetilde{\Omega}_2^- \cup \widetilde{\Omega}_3^- \cup \mathbb{D}(\bar{a}_6, \varepsilon_6) \setminus \overline{\mathbb{D}}$ , except the subsets of the unit circle and the real axis, consists of  $\ell_3 : \zeta = t e^{-0.72\pi i} (t \geq t_3)$ ;  $\ell_4 : \zeta = t - 2.7i (t_4 \leq$



$t \leq x_E$ );  $\ell_5 : \zeta = x_E + it$  ( $-2.7 \leq t \leq t_5$ );  $\ell_6 : \zeta = \zeta(r_3 e^{\theta\pi i})$  ( $-0.5 \leq \theta \leq -0.4$ );  $\ell_7 : \zeta = \zeta(r e^{-0.4\pi i})$  ( $r_2 \leq r \leq r_3$ );  $\ell_8 : \zeta = \zeta(r_2 e^{\theta\pi i})$  ( $-0.4 \leq \theta \leq \theta_6$ );  $\ell_9 : \zeta = \bar{a}_6 + \varepsilon_6 e^{\theta\pi i}$  ( $\theta_7 \leq \theta \leq \theta_8$ );  $\ell_{10} : \zeta = \bar{a}_6 + \varepsilon_6 e^{\theta\pi i}$  ( $\theta_9 \leq \theta \leq \theta_{10}$ ) and  $\ell_{11} : \zeta = \zeta(r_1 e^{\theta\pi i})$  ( $\theta_{11} \leq \theta \leq 0$ ); where

$$\begin{aligned} t_3 &= \frac{2.7}{\sin(0.28\pi)}, & t_4 &= -\frac{2.7}{\tan(0.28\pi)}, & t_5 &= \operatorname{Im} \zeta(-r_3 i) (\doteq -1.5971\dots); \\ \theta_6 &(\doteq -0.3488\dots), & \theta_7 &(\doteq -0.9513\dots), & \theta_8 &(\doteq -0.5711\dots), \\ \theta_9 &(\doteq 0.1688\dots), & \theta_{10} &(\doteq 0.3800\dots), & \theta_{11} &(\doteq -0.1770\dots); \end{aligned} \quad (3.73^*)$$

such that  $|\zeta(r_2 e^{\theta_6\pi i}) - \bar{a}_6| = \varepsilon_6$ ,  $\zeta(r_2 e^{\theta_6\pi i}) = \bar{a}_6 + \varepsilon_6 e^{\theta_6\pi i}$ ,  $|\bar{a}_6 + \varepsilon_6 e^{\theta_7\pi i}| = 1$ ,  $|\bar{a}_6 + \varepsilon_6 e^{\theta_{10}\pi i}| = 1$ ,  $\bar{a}_6 + \varepsilon_6 e^{\theta_9\pi i} = \zeta(r_1 e^{\theta_{11}\pi i})$  and  $|\zeta(r_1 e^{\theta_{11}\pi i}) - \bar{a}_6| = \varepsilon_6$ .

We first prove that  $Q(\ell_i) \subset \mathbb{C} \setminus \overline{W}_0$  for  $3 \leq i \leq 8$  and  $i = 11$ . In the following, we denote  $u' = \frac{5.3}{\cos(\pi/5)}$ . For  $\ell_3$ , since it is unbounded, we divide it into two parts for consideration. If  $\zeta = t e^{-0.72\pi i}$  for  $t \geq 100$ , similar to (3.68\*), we have

$$\begin{aligned} |Q(\zeta) - \zeta - b_0| &< \max_{|\zeta|=100} |Q(\zeta) - \zeta - b_0| (\doteq 0.2166\dots) \\ &< (100 \sin(0.02\pi) + u' - b_0) \cos(\frac{\pi}{5}) (\doteq 4.4355\dots). \end{aligned} \quad (3.74^*)$$

This means that the image of  $\zeta = t e^{-0.72\pi i}$  ( $t \geq 100$ ) under  $Q$  is contained in  $\mathbb{C} \setminus \overline{W}_0$ . For the part of  $t_3 \leq t \leq 100$ , we have

$$\begin{aligned} \max_{t \in [t_3, 100]} \operatorname{Re} Q(t e^{-0.72\pi i}) &(\doteq 1.2378\dots) < 2, \text{ and} \\ \max_{t \in [t_3, 100]} \arg(Q(t e^{-0.72\pi i}) - u') &(\doteq -2.2573\dots) < -\frac{7\pi}{10} (\doteq -2.1991\dots). \end{aligned} \quad (3.75^*)$$

Therefore, we have  $Q(\ell_3) \subset \mathbb{C} \setminus \overline{W}_0$ . By

$$\begin{aligned} \max_{t \in [t_4, x_E]} \operatorname{Re} Q(t - 2.7i) &(\doteq 1.7079\dots) < 2, \text{ and} \\ \max_{t \in [t_4, x_E]} |\operatorname{Im} Q(t - 2.7i)| &(\doteq 4.7446\dots) < (u' - 2) \tan \frac{3\pi}{10} (\doteq 6.2641\dots), \end{aligned} \quad (3.76^*)$$

we have  $Q(\ell_4) \subset \mathbb{C} \setminus \overline{W}_0$ . By

$$\begin{aligned} \max_{t \in [-2.7, t_5]} \operatorname{Re} Q(x_E + it) &(\doteq 1.7079\dots) < 2, \text{ and} \\ \max_{t \in [-2.7, t_5]} |\operatorname{Im} Q(x_E + it)| &(\doteq 5.1552\dots) < (u' - 2) \tan \frac{3\pi}{10} (\doteq 6.2641\dots), \end{aligned} \quad (3.77^*)$$

we have  $Q(\ell_5) \subset \mathbb{C} \setminus \overline{W}_0$ . By

$$\begin{aligned} \max_{\theta \in [-0.5, -0.4]} \operatorname{Re} Q(\zeta(r_3 e^{\theta\pi i})) &(\doteq -6.28317\dots) < 2, \text{ and} \\ \max_{\theta \in [-0.5, -0.4]} |\operatorname{Im} Q(\zeta(r_3 e^{\theta\pi i}))| &(\doteq 11.6393\dots) < (u' + 6) \tan \frac{3\pi}{10} (\doteq 17.2752\dots), \end{aligned} \quad (3.78^*)$$

we have  $Q(\ell_6) \subset \mathbb{C} \setminus \overline{W}_0$ . By

$$\begin{aligned} \max_{r \in [r_2, r_3]} \operatorname{Re} Q(\zeta(r e^{-0.4\pi i})) &(\doteq -32.1938\dots) < 2, \text{ and} \\ \max_{r \in [r_2, r_3]} |\operatorname{Im} Q(\zeta(r e^{-0.4\pi i}))| &(\doteq 11.6393\dots) < (u' + 32) \tan \frac{3\pi}{10} (\doteq 53.0611\dots), \end{aligned} \quad (3.79^*)$$

we have  $Q(\ell_7) \subset \mathbb{C} \setminus \overline{W}_0$ . By

$$\begin{aligned} \max_{\theta \in [-0.4, \theta_6]} \operatorname{Re} Q(\zeta(r_2 e^{\theta\pi i})) &(\doteq -45.4566\dots) < 2, \text{ and} \\ \max_{\theta \in [-0.4, \theta_6]} |\operatorname{Im} Q(\zeta(r_2 e^{\theta\pi i}))| &(\doteq 60.8537\dots) < (u' + 45) \tan \frac{3\pi}{10} (\doteq 70.9541\dots), \end{aligned} \quad (3.80^*)$$

we have  $Q(\ell_8) \subset \mathbb{C} \setminus \overline{W}_0$ . By

$$\begin{aligned} \max_{\theta \in [\theta_{11}, 0]} \operatorname{Re} Q(\zeta(r_1 e^{\theta \pi i})) (\doteq 0.7179 \dots) &< 2, \text{ and} \\ \max_{\theta \in [\theta_{11}, 0]} \arg(Q(\zeta(r_1 e^{\theta \pi i})) - u') (\doteq -2.7447 \dots) &< -\frac{7\pi}{10} (\doteq -2.1991 \dots). \end{aligned} \quad (3.81^*)$$

we have  $Q(\ell_{11}) \subset \mathbb{C} \setminus \overline{W}_0$ . Therefore, we have proved that  $Q(\ell_{11}) \cup \bigcup_{i=3}^8 Q(\ell_i) \subset \mathbb{C} \setminus \overline{W}_0$ .

For  $\ell_9$  and  $\ell_{10}$ , we will divide them into two parts for discussion. By

$$\begin{aligned} \max_{\theta \in [-0.65, \theta_8]} \operatorname{Re} Q(\bar{a}_6 + \varepsilon_6 e^{\theta \pi i}) (\doteq -107.636 \dots) &< 2, \text{ and} \\ \max_{\theta \in [-0.65, \theta_8]} |\operatorname{Im} Q(\bar{a}_6 + \varepsilon_6 e^{\theta \pi i})| (\doteq 60.8537 \dots) &< (u' + 107) \tan \frac{3\pi}{10} (\doteq 156.29 \dots), \end{aligned} \quad (3.82^*)$$

we have  $Q(\ell'_9) \subset \mathbb{C} \setminus \overline{W}_0$ , where  $\ell'_9 = \{\bar{a}_6 + \varepsilon_6 e^{\theta \pi i} : -0.65 \leq \theta \leq \theta_8\}$ . On the other hand, a numerical estimation shows that

$$\min_{\theta \in [\theta_7, -0.65]} \operatorname{Re} \sqrt{Q(\bar{a}_6 + \varepsilon_6 e^{\theta \pi i})} (\doteq 0.4320 \dots) > 0. \quad (3.83^*)$$

This means that  $Q(\zeta) \cap (-\infty, 0] = \emptyset$  if  $\zeta \in \ell''_9 := \{\bar{a}_6 + \varepsilon_6 e^{\theta \pi i} : \theta_7 \leq \theta \leq -0.65\}$ . Note that  $(-\infty, 0) = \Gamma_c^Q$ . Hence  $\ell''_9$  is disjoint with any  $\gamma_{ci}$ . In particular,  $\ell''_9 \cap \gamma_{c3} = \emptyset$ . Since  $|\bar{a}_6 + \varepsilon_6 e^{\theta_7 \pi i}| = 1$  and  $\bar{a}_6 + \varepsilon_6 e^{\theta_7 \pi i}$  is contained in  $\partial U_-^Q$ , hence  $\ell''_9$  is contained in  $\overline{U}_{5-} \cup U_-^Q$ . Similarly, by

$$\begin{aligned} \max_{\theta \in [\theta_9, 0.23]} \operatorname{Re} Q(\bar{a}_6 + \varepsilon_6 e^{\theta \pi i}) (\doteq -121.72 \dots) &< 2, \text{ and} \\ \max_{\theta \in [\theta_9, 0.23]} |\operatorname{Im} Q(\bar{a}_6 + \varepsilon_6 e^{\theta \pi i})| (\doteq 55.462 \dots) &< (u' + 121) \tan \frac{3\pi}{10} (\doteq 175.5591 \dots), \\ \min_{\theta \in [0.23, \theta_{10}]} \operatorname{Re} \sqrt{Q(\bar{a}_6 + \varepsilon_6 e^{\theta \pi i})} (\doteq 2.3197 \dots) &> 0. \end{aligned} \quad (3.84^*)$$

we have  $Q(\ell'_{10}) \subset \mathbb{C} \setminus \overline{W}_0$  and  $\ell'_{10}$  is contained in  $\overline{U}'_{5+}$ , where  $\ell'_{10} = \{\bar{a}_6 + \varepsilon_6 e^{\theta \pi i} : \theta_9 \leq \theta \leq 0.23\}$  and  $\ell''_{10} = \{\bar{a}_6 + \varepsilon_6 e^{\theta \pi i} : 0.23 \leq \theta \leq \theta_{10}\}$ .

By the partition on the complex plane of  $Q$  in §3.4, we know that  $Q : \mathcal{U}_{123} \rightarrow \mathbb{C} \setminus (-\infty, 0]$  is a branched covering of degree 3 branched over  $cv$ . Since  $\partial W_0$  is unbounded and does not contain the critical value  $cv$ , it follows that  $Q^{-1}(\partial W_0) \cap \overline{\mathcal{U}}_{123}$  is a unbounded simple curve passing through  $c_+^Q$  and  $c_-^Q$  (compare Figure 12). Note that  $Q(\ell'_9) \cup Q(\ell'_{10}) \cup Q(\ell_{11}) \cup \bigcup_{i=3}^8 Q(\ell_i) \subset \mathbb{C} \setminus \overline{W}_0$  and  $\ell''_9 \cup \ell''_{10} \subset \partial \mathbb{D}(\bar{a}_6, \varepsilon_6)$ . Therefore, we have  $Q^{-1}(W_0) \cap \mathcal{U}_{123} \cap \mathbb{H}^- \subset \tilde{\Omega}_0^- \cup \tilde{\Omega}_1^- \cup \tilde{\Omega}_2^- \cup \tilde{\Omega}_3^- \cup \mathbb{D}(\bar{a}_6, \varepsilon_6)$ . By the symmetry, we have  $Q^{-1}(W_0) \cap \mathcal{U}_{123} \subset \tilde{\Omega}_0 \cup \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \tilde{\Omega}_3 \cup \mathbb{D}(a_6, \varepsilon_6) \cup \mathbb{D}(\bar{a}_6, \varepsilon_6)$ .

(d) By the symmetry, it is sufficient to prove that  $\ell := (\partial \mathbb{D}(\bar{a}_5, \varepsilon_5) \setminus \mathbb{D}) \cap \mathbb{H}^-$  is on the left of  $Q^{-1}(\partial W_0) \cap \overline{\mathcal{U}}_{123}$ . Actually, the arc  $\ell$  has been parameterized in the proof of Lemma 3.22 (b) such that its closure can be written as  $\bar{\ell} : \zeta = \bar{a}_5 + \varepsilon_5 e^{\theta \pi i} (\theta_3 \leq \theta \leq \theta_4)$ , where  $\theta_3$  and  $\theta_4$  are defined in (3.22\*). It is known from (3.24\*) that  $\ell' := \{\bar{a}_5 + \varepsilon_5 e^{\theta \pi i} : \theta_3 \leq \theta \leq -0.55\}$  is contained in  $\overline{U}'_{5+}$ . On the other hand, we have

$$\begin{aligned} \max_{\theta \in [-0.55, \theta_4]} \operatorname{Re} Q(\bar{a}_5 + \varepsilon_5 e^{\theta \pi i}) (\doteq 1.5813 \dots) &< 2, \text{ and} \\ \min_{\theta \in [-0.55, \theta_4]} |\arg(Q(\bar{a}_5 + \varepsilon_5 e^{\theta \pi i}) - u')| (\doteq 2.3615 \dots) &> \frac{7\pi}{10} (\doteq 2.1991 \dots). \end{aligned} \quad (3.85^*)$$

This means that  $Q(\tilde{\ell}) \subset \mathbb{C} \setminus \overline{W}_0$ , where  $\tilde{\ell} := \{\bar{a}_5 + \varepsilon_5 e^{\theta \pi i} : -0.55 \leq \theta \leq \theta_4\}$ . Combine the results on  $\ell'$  and  $\tilde{\ell}$ , it follows that  $\bar{\ell} = \ell' \cup \tilde{\ell}$  is on the left of  $Q^{-1}(\partial W_0) \cap \overline{\mathcal{U}}_{123}$ . Hence if  $\zeta \in Q^{-1}(W_0) \cap (\overline{\mathcal{U}}_{2+} \cup \overline{\mathcal{U}}_{3-})$ , then  $\zeta \in (\overline{\mathcal{U}}_{2+} \cup \overline{\mathcal{U}}_{3-}) \setminus (\mathbb{D}(a_5, \varepsilon_5) \cup \mathbb{D}(\bar{a}_5, \varepsilon_5))$ .  $\square$

### 3.13. Construction of $\Psi_1$ – Relating $D_n$ 's to $P$ .

*Proof of Proposition 3.8.* By Proposition 3.7, the sets  $\overline{D}_0, \overline{D}'_0, \overline{D}''_0, \overline{D}_{-1}, \overline{D}'''_{-1}, \overline{D}''''_{-1}$  and  $\overline{D}^\sharp_0$  are contained in  $\pi_X(X_{1+} \cup X_{2-})$ . Hence they can also be regarded as subsets of  $X_{1+} \cup X_{2-}$ . For  $n = 1, 2, \dots$ , define

$$\begin{aligned} D_{-n-1} &:= g^n(D_{-1}); \quad D'_{-n} := g^n(D'_0); \quad D''_{-n} := g^n(D''_0); \\ D'''_{-n-1} &:= g^n(D'''_{-1}); \quad D''''_{-n-1} := g^n(D''''_{-1}); \quad D^\sharp_{-n} := g^n(D^\sharp_0). \end{aligned}$$

The Fatou coordinate  $\Phi_{attr}$  extends naturally to  $\tilde{\Phi}_{attr}$  on these domains and their closure. Let

$$D = \{z : 0 < \operatorname{Re} z < 1 \text{ and } |\operatorname{Im} z| < \eta\} \text{ and } D^\sharp = \{z : 0 < \operatorname{Re} z < 1 \text{ and } \operatorname{Im} z > \eta\}.$$

As in [IS08, §5.M], we name the boundary segments of  $D$  and  $D^\sharp$  as follows (see left picture in Figure 13):

$$\begin{aligned} \partial^l_+ D &= 0 + [0, \eta]i; \quad \partial^l_- D = 0 + [0, -\eta]i; \quad \partial^r_+ D = 1 + [0, \eta]i; \quad \partial^r_- D = 1 + [0, -\eta]i; \\ \partial^h_+ D &= \partial^h D^\sharp = [0, 1] + \eta i; \quad \partial^h_- D = [0, 1] - \eta i; \quad \partial^l D^\sharp = 0 + [\eta, +\infty)i; \quad \partial^r D^\sharp = 1 + [\eta, +\infty)i. \end{aligned}$$

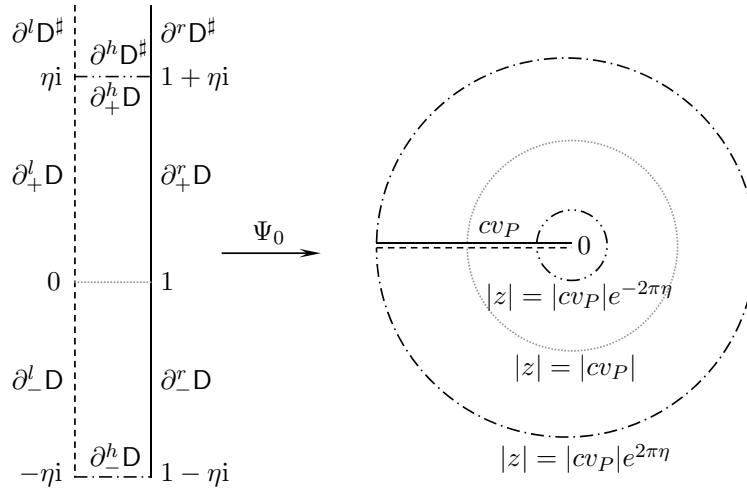


Figure 13: The domain and the range of  $\Psi_0(z) = cv_P e^{2\pi i z}$ . These two pictures are just topologically correct but not conformally precise.

Note that  $\Phi_{attr}(z) - 1$  maps  $D_1$  and  $D_1^\sharp$  homeomorphically onto  $D$  and  $D^\sharp$  including their boundaries respectively. Hence we can name the boundary segments of  $D_1$  and  $D_1^\sharp$  by  $\partial^l_+ D_1$  and  $\partial^h D_1^\sharp$ , etc. according to their images under  $\Phi_{attr}(z) - 1$ . In the following, the same naming convention will be applied to domains that are mapped homeomorphically onto  $D_1$  and  $D_1^\sharp$  by the iterates of  $F$  or by  $Q$ .

The following lemma is an extended version of [IS08, Lemma 5.34]. We omit the proof here since the idea of proof is almost the same (compare Figure 5 and [IS08, Figure 8]).

**Lemma 3.38.** (a)  $g(D_0) = D_{-1}$  and  $g(D_1^\sharp) = D_0^\sharp$ .

(b) Among the closed domains  $\{\bar{D}_n, \bar{D}'_n, \bar{D}''_n, \bar{D}'''_n, \bar{D}''''_n, \bar{D}^\sharp_n \mid n = 0, -1, -2, \dots\}$ , the intersecting pairs are exactly as follows:

$$\left\{ \begin{array}{l} \bar{D}_n \cap \bar{D}_{n-1} = \partial_+^l D_n = \partial_+^r D_{n-1}, \quad \bar{D}_{n-1} \cap \bar{D}'_n = \partial_-^r D_{n-1} = \partial_-^l D'_n \\ \bar{D}'_n \cap \bar{D}'''_{n-1} = \partial_+^l D'_n = \partial_+^r D'''_{n-1}, \quad \bar{D}'''_{n-1} \cap \bar{D}''_n = \partial_-^r D'''_{n-1} = \partial_-^l D''_n, \\ \bar{D}''_n \cap \bar{D}''''_{n-1} = \partial_+^l D''_n = \partial_+^r D''''_{n-1}, \quad \bar{D}''''_{n-1} \cap \bar{D}_n = \partial_-^r D''''_{n-1} = \partial_-^l D_n, \\ \bar{D}_n \cap \bar{D}'_n \cap \bar{D}''_n = \bar{D}_{n-1} \cap \bar{D}'''_{n-1} \cap \bar{D}''''_{n-1} = \bar{D}_n \cap \bar{D}'_n = \bar{D}_n \cap \bar{D}''_{n-1} = \bar{D}_n \cap \bar{D}''_n \\ = \bar{D}_{n-1} \cap \bar{D}'''_{n-1} = \bar{D}_{n-1} \cap \bar{D}''_n = \bar{D}_{n-1} \cap \bar{D}''''_{n-1} \\ = \bar{D}'_n \cap \bar{D}''_n = \bar{D}'_n \cap \bar{D}''''_{n-1} = \bar{D}'''_{n-1} \cap \bar{D}''''_{n-1} = a \text{ point}, \\ \bar{D}_n \cap \bar{D}^\sharp_n = \partial_+^h D_n = \partial^h D_n^\sharp, \quad \bar{D}^\sharp_n \cap \bar{D}^\sharp_{n-1} = \partial^l D_n^\sharp = \partial^r D_{n-1}^\sharp, \\ \bar{D}_n \cap \bar{D}^\sharp_{n-1} = \bar{D}_{n-1} \cap \bar{D}^\sharp_n = a \text{ point}. \end{array} \right. \quad (3.86)$$

Let

$$\begin{aligned} \mathcal{U} &= \mathcal{U}_{1+}^P \cup \mathcal{U}_{1-}^P \cup \gamma_{c1}^P, \quad \mathcal{U}' = \mathcal{U}_{2-}^P \cup \mathcal{U}_{4+}^P \cup \gamma_{c4}^P, \quad \mathcal{U}'' = \mathcal{U}_{3-}^P \cup \mathcal{U}_{5+}^P \cup \gamma_{c5}^P, \\ \mathcal{U}''' &= \mathcal{U}_{2+}^P \cup \mathcal{U}_{4-}^P \cup \gamma_{c2}^P, \quad \mathcal{U}'''' = \mathcal{U}_{3+}^P \cup \mathcal{U}_{5-}^P \cup \gamma_{c3}^P. \end{aligned}$$

Each of these five domains is mapped homeomorphically by  $P$  onto  $\mathbb{C} \setminus (-\infty, 0]$ . The map  $\Psi_0(z) = cv_P e^{2\pi iz} = -\frac{64}{225\sqrt{5}} e^{2\pi iz}$  defined in Proposition 3.8 maps the rectangle  $D = \{z : 0 < \operatorname{Re} z < 1 \text{ and } |\operatorname{Im} z| < \eta\}$  onto  $\{z : |cv_P| e^{-2\pi\eta} < |z| < |cv_P| e^{2\pi\eta}\} \setminus (-\infty, 0]$  and maps the half-infinite strip  $D^\sharp = \{z : 0 < \operatorname{Re} z < 1 \text{ and } \operatorname{Im} z > \eta\}$  onto  $\{z : |z| < |cv_P| e^{-2\pi\eta}\} \setminus (-\infty, 0]$  (see Figure 13).

We now define  $\Psi_1$  in the interior of  $D_n$  etc as following, where  $n \leq 0$ :

$$\Psi_1 = \begin{cases} (P|_{\mathcal{U}})^{-1} \circ \Psi_0 \circ \tilde{\Phi}_{attr} & \text{on } D_n \cup D_n^\sharp \\ (P|_{\mathcal{U}'})^{-1} \circ \Psi_0 \circ \tilde{\Phi}_{attr} & \text{on } D'_n \\ (P|_{\mathcal{U}''})^{-1} \circ \Psi_0 \circ \tilde{\Phi}_{attr} & \text{on } D''_n \\ (P|_{\mathcal{U}''''})^{-1} \circ \Psi_0 \circ \tilde{\Phi}_{attr} & \text{on } D'''_{n-1} \\ (P|_{\mathcal{U}''''})^{-1} \circ \Psi_0 \circ \tilde{\Phi}_{attr} & \text{on } D''''_{n-1}. \end{cases}$$

Then  $\Psi_1$  is a homeomorphism from each domain onto its image, and it extends continuously to the closure of each domain. Note that  $\Psi_1$  is holomorphic in the interior of each domain since  $\Psi_1$  is a branch of  $P^{-1} \circ \Psi_0 \circ \tilde{\Phi}_{attr}$  on each domain. In order to prove  $\Psi_1$  is holomorphic on the whole domain, we just need to show that on the common boundary of any two domains above, the two extensions are consistent.

Now we check the consistent condition according to (3.86). If  $z \in D_n$  tends to  $\partial_+^l D_n$ , then  $\Psi_0 \circ \tilde{\Phi}_{attr}(z)$  tends to  $[cv_P, 0) = \Gamma_a^P \cup \{cv_P\}$  from the lower side. Hence  $\Psi_1(z) \in \mathcal{U}$  tends to  $[cp_P, 0) = \gamma_{a1}^P$  from the lower side (see Figure 13). If  $z \in D_{n-1}$  tends to  $\partial_+^r D_{n-1}$  from another side, then  $\Psi_0 \circ \tilde{\Phi}_{attr}(z)$  tends to  $[cv_P, 0) = \Gamma_a^P \cup \{cv_P\}$  from the upper side, and  $\Psi_1(z) \in \mathcal{U}$  tends to  $[cp_P, 0) = \gamma_{a1}^P$  from the upper side. The map  $\Psi_1$  matches completely along  $\bar{D}_n \cap \bar{D}_{n-1} = \partial_+^l D_n = \partial_+^r D_{n-1}$  since  $P$  is a homeomorphism in a neighborhood of each  $\gamma_{ai}^P$ . This means that  $\Psi_1$  is holomorphic there. Similarly, one can check the rest of (3.86) easily. Therefore, the map  $\Psi_1$  is defined on  $U =$  the interior of  $\bigcup_{n=-\infty}^0 (\bar{D}_n \cup \bar{D}'_n \cup \bar{D}''_n \cup \bar{D}'''_{n-1} \cup \bar{D}''''_{n-1} \cup \bar{D}_n^\sharp)$ . Now it is obviously that  $P \circ \Psi_1 = \Psi_0 \circ \tilde{\Phi}_{attr}$  and  $\Psi_1 : U \rightarrow U_\eta^P \setminus \{0\} = V' \setminus \{0\}$  is a surjection. The assertions (b), (c) and (d) in Proposition 3.8 now is straightforward. Finally, the holomorphic dependence in (e) can be proved as in [IS08] using a word for word copy. We omit the details.  $\square$

**3.14. Remarks on the constants.** As pointed out in [IS08, §5.N], the important constants are  $\eta$ ,  $\rho$ ,  $R$  and  $r_1$ . The number  $\eta$  is determined by (3.60\*) and (3.61\*). There,  $\eta$  can be chosen between 3.1 and 5.4. By (3.61\*), the upper bound of  $\eta$  is effected by  $R_1$  but actually, it is effected by  $R$  since  $R_1$  and  $R$  only need to satisfy (3.63) (i.e.  $R_1 + cv \leq R$ ). However,  $R$  cannot be too large, the upper bound of  $R$  is determined by Lemma 3.24 since  $E_{r_1}$  needs to be covered by eight disks and some of them are affected by  $R$ . Similarly,  $\rho$  cannot be too small by Lemma 3.24 and cannot be too large by Lemma 3.29. In fact, Lemmas 3.24 and 3.29 also show that the constant  $r_1$  cannot be too large or too small.

The disk  $\mathbb{D}(a_5, \varepsilon_5)$  (and hence  $\mathbb{D}(\bar{a}_5, \varepsilon_5)$ ) is also important. Lemma 3.24 (c) shows that  $\mathbb{D}(a_5, \varepsilon_5)$  cannot be too small but Lemma 3.36 (d) suggests that  $\mathbb{D}(a_5, \varepsilon_5)$  cannot be too large and the position of  $\mathbb{D}(a_5, \varepsilon_5)$  must be located suitably.

The constant  $u_2 = 5.4$  defined at the beginning of §3.11 is determined by Lemma 3.32 and also Lemma 3.27 (b).

In Table 1, there are many constants that we give the approximate value up to the four digits after the decimal point. Most of them are effected by  $\sigma$ ,  $\mu$ ,  $\nu$  and  $\kappa$ .

#### 4. THE STRUCTURES OF THE RENORMALIZATION

In this section, we will use the languages of structure and sub-structure to analyze the parabolic renormalization with local degree three. These languages were first defined in [Ché14].

**4.1. Structures and sub-structures.** The maps in the class  $\mathcal{F}_1$  defined in the introduction are not branched coverings because of the special choice of  $V$ . In fact, this class comes from another class of maps with much richer branched covering structure, which is also invariant under parabolic renormalization (see [Ché14]). In order to restate the result in the Main Theorem, we first give the definitions of structure and substructure.

**Definition (Structure).** Let  $X_1$ ,  $X_2$  and  $Y$  be Riemann surfaces. Assume that  $a : I \rightarrow \{a_i \in X_1 : i \in I\}$  and  $b : I \rightarrow \{b_i \in X_2 : i \in I\}$  are two marked maps with collections of marked points in  $X_1$  and  $X_2$  respectively, where  $I$  is an index set. Consider two analytic maps which are nowhere locally constant  $f_1 : X_1 \rightarrow Y$  and  $f_2 : X_2 \rightarrow Y$ . The pairs  $(a, f_1)$  and  $(b, f_2)$  are called *structurally equivalent* if there exists an analytic isomorphism  $\phi : X_1 \rightarrow X_2$  such  $f_1 = f_2 \circ \phi$  and  $b = \phi \circ a$ . It is easy to verify that structurally equivalent is an equivalence relation and this relation depends only on  $I$  and  $Y$ . For emphasis, the terminology  $(I, Y)$ -*structurally equivalent* is also used. The equivalence classes are called *structures* (or  $(I, Y)$ -*structures*).

**Definition (Sub-structure).** With the same definitions and notations as above, but assuming  $\phi$  is an analytic injection instead of analytic isomorphism (i.e. not necessary surjective), we will say that the structure of  $(a, f_1)$  is a *sub-structure* of that of  $(b, f_2)$ . Sometimes we also call that  $(b, f_2)$  has at least the structure  $(a, f_1)$ .

The restriction of partial coverings (containing the marked points) induces a pre-order on structures. It was proved in [Ché14] that the pre-order is not always an order but it is an order if one considers the subclass of structures with connected  $X$  and at least one marked point.

Let's focus on the structure and sub-structure of parabolic renormalization with local degree three. For the general statement, see [Ché14]. Define

$$B_3(z) = \left( \frac{z + 1/2}{1 + z/2} \right)^3.$$

Then  $B_3$  is a Blaschke product whose Julia set is the unit circle. The point  $z = 1$  is a 1-parabolic fixed point with two attracting petals. In particular, the unit disk  $\mathbb{D}$  and  $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$

are two immediate basins of 1. For the following theorem, one can find a similar statement in [DH84-85, exposé IX].

**Theorem 4.1.** *Let  $f : U \subset \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a holomorphic map with a 1-parabolic fixed point  $z_0 \in U$ . Suppose that the immediate basin  $A$  of  $z_0$  is fixed by  $f$  and compactly contained in  $U$  and contains only one critical point with local degree 3. Then the restriction of  $f$  to  $A$  is analytically conjugated to the restriction of  $B_3$  to  $\mathbb{D}$ .*

Let  $\Phi_{attr}[B_3] : \mathbb{D} \rightarrow \mathbb{C}$  be the extended attracting Fatou coordinate<sup>14</sup> for one of the immediate basins  $\mathbb{D}$  of  $B_3$ . Note that this map has two repelling petals with vertical axes. We choose the top one and let  $\Psi_{rep}[B_3]$  denote the corresponding extended repelling Fatou coordinate. Moreover, let  $h[B_3] := \Phi_{attr} \circ \Psi_{rep}$  be the extended horn map.

Theorem 4.1 has many consequences. The following results are discovered by Shishikura, Lanford and Yampolsky (see [LY14]). They also have been stated in [Ché14].

**Corollary 4.2** (Shishikura, Landord and Yampolsky). *With the same definitions and notations as in Theorem 4.1, if  $\xi : A \rightarrow \mathbb{D}$  is the conjugacy such that  $B_3 \circ \xi = \xi \circ f$ , then there exists a constant  $c \in \mathbb{C}$  depending only the normalization of the Fatou coordinates such that  $\Phi_{attr}[B_3] \circ \xi = \Phi_{attr}[f] + c$  whenever both sides are defined.*

Therefore, in the language of structure, it follows from Corollary 4.2 that  $\Phi_{attr}[f] + c$  is structurally equivalent to  $\Phi_{attr}[B_3]$  over  $\mathbb{C}$  since they only differ a conformal isomorphism  $\xi$ .

**Theorem 4.3** (Shishikura, Landord and Yampolsky). *All the maps in Theorem 4.1 with non-degenerate 1-parabolic fixed points (i.e. with only one petal) have structurally equivalent upper parabolic renormalizations, when the latter is normalized by setting the singular values to 0, 1 and  $\infty$ . More precisely they are  $(I, Y)$ -structurally equivalent with  $Y = \widehat{\mathbb{C}}$ ,  $I$  being a singleton and the marked point being the origin. The same holds for the lower renormalization, and the upper one is structurally equivalent to the conjugate of the lower by the reflection  $z \mapsto 1/\bar{z}$ . Moreover the upper or lower parabolic renormalization is defined on a simply connected set and has exactly 3 singular values: the asymptotic values 0, 1 and one critical value.*

See Figure 14 and the text following for the explanation.

Figure 14 shows the cubic case when the maps are  $f(z) = z^3 + \frac{2}{3\sqrt{3}}$  and  $B_3$ . These six pictures are ordered in a rectangle such that the pictures in the first column indicate the dynamical chessboards of  $f$  atop and of  $B_3$  below. The next column represents views of their chessboards in repelling Fatou coordinates. The last column is the projection to  $\mathbb{C}^*$  of the middle column by the exponential map  $\text{Exp}^\sharp$ . The vertical arrows are structure isomorphisms for the following respective maps (properly normalized): the attracting Fatou coordinate for the first row, the horn map for the second row and the parabolic renormalization for the last one. Note that the tiny loops in the last column are the images of the big unbounded domain that lie above in both middle pictures.

Recall that  $\mathcal{F}_0$  is a class defined in §3.1. As an immediate corollary of Theorem 4.3, we have:

**Theorem 4.4.**  $\mathcal{R}_0(\mathcal{F}_0) \subset \mathcal{F}_0$ . *Moreover, any map in  $\mathcal{R}_0(\mathcal{F}_0)$  can be written as  $\mathcal{R}_0(B_3) \circ \varphi^{-1}$ , where  $\mathcal{R}_0(B_3)$  is defined on  $\mathbb{D}$  and  $\varphi : \mathbb{D} \rightarrow \mathbb{C}$  is a univalent mapping satisfying  $\varphi(0) = 0$  and  $\varphi'(0) = 1$ .*

<sup>14</sup>The attracting and repelling Fatou coordinates  $\Phi_{attr}$  and  $\Phi_{rep}$  are only defined for the maps with non-degenerate 1-parabolic fixed points in §2. However, the similar definition can be given for the degenerate case, i.e. with more than one petal. The attracting Fatou coordinate  $\Phi_{attr}$  can be extended to the maximal region: to whole attracting basin. But for the repelling Fatou coordinate  $\Phi_{rep}$ , there is no similar maximal extension. Instead, there exists a unique maximal extension of the reciprocal  $\Psi_{rep} := \Phi_{rep}^{-1}$  such that  $\Psi_{rep}(z) + 1 = B_3 \circ \Psi_{rep}(z)$ . For more details, one can refer, for example, [BE02], [DSZ98] and [Zin97].

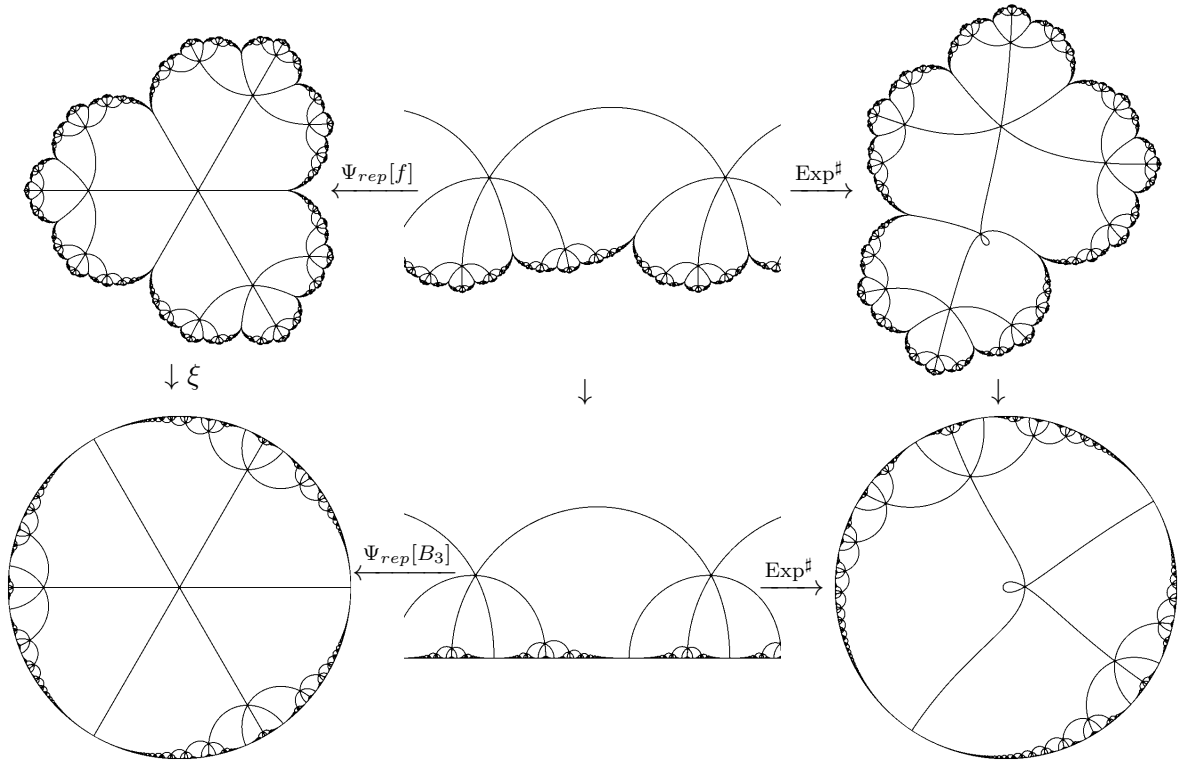


Figure 14: Illustration of Theorem 4.3 for  $f(z) = z^3 + \frac{2}{3\sqrt{3}}$ , where  $f$  has a 1-parabolic fixed point at  $\frac{1}{\sqrt{3}}$  whose immediate basin contains the critical point 0 with local degree 3.

By Theorem 1.1, this means that the parabolic renormalization  $\mathcal{R}_0$  on  $\mathcal{F}_1$  has a unique fixed point which belongs to  $\mathcal{F}_0$ .

**4.2. Visualizing structures of  $\mathcal{V}$  and  $\mathcal{V}'$  in degree three.** For a given structure  $\mathcal{V}'$  and a sub-structure  $\mathcal{V}$ , we call that  $\mathcal{V}$  is a *relatively compact* sub-structure of  $\mathcal{V}'$  if the maps in  $\mathcal{V}$  are structurally equivalent to restrictions of maps in  $\mathcal{V}'$  to relatively compact open subsets of their domains (not just subsets).

Before restating the Main Theorem in the language of structure and substructure, we first give the explicit description of a structure  $\mathcal{V}'$ . We define a Riemann surface with a natural projection over  $\mathbb{C}/\mathbb{Z}$  as follows: cut the cylinder  $\mathbb{C}/\mathbb{Z}$  so as to obtain only the part where  $\text{Im } z > -\eta$  with  $\eta = 3.1$ . Then slit the cylinder along the vertical segment from 0 to  $-\eta i$ . To obtain this, we glue two same rectangles  $\{z \in \mathbb{C} : \text{Re } z \in (-1, 1) \text{ and } \text{Im } z \in (-\eta, \eta)\}$ , cut along the same segments. Finally, we glue each side of the segment in one piece to the opposite side on the other piece. See Figure 15.

The structure  $\mathcal{V}$  is obtained by mapping conformally the domain of  $f$  minus the origin to the complement of the closed unit disk and removing the interior of the ellipse  $E$ . Now we can restate the Main Theorem in the language of structure and substructure.

**Theorem 4.5.** *Let  $I$  be a singleton and  $Y = \widehat{\mathbb{C}}$ . There exists an explicit pair of  $(I, Y)$ -structures  $\mathcal{V}$  and  $\mathcal{V}'$  with the following properties:*

- (a)  $\mathcal{V}$  is a relatively compact sub-structure of  $\mathcal{V}'$  and  $\mathcal{V}'$  is a sub-structure of the universal structure of Theorem 4.3;
- (b) For any  $(a, f) \in \mathcal{V}$ , the map  $f$  is defined on a connected and simply connected Riemann surface and has exactly one critical point with local degree 3; the same holds for  $\mathcal{V}'$ ;

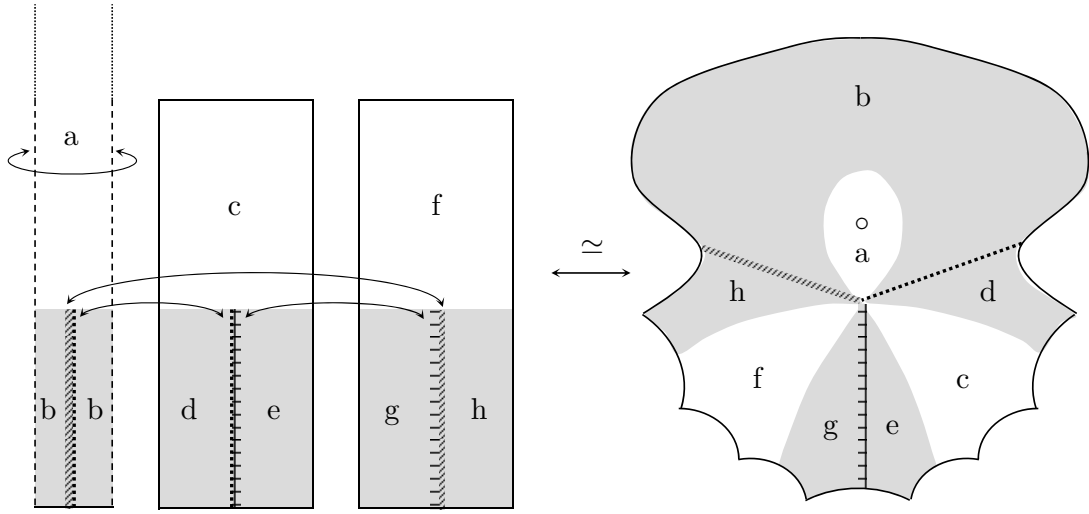


Figure 15: The structure  $\mathcal{V}'$ . Only the left picture is conformally correct. See also the left picture in Figure 16, which accurately shows how  $\mathcal{V}'$  sits as a substructure of the structure of  $\mathcal{R}_0(B_3)$ .

- (c) For any map in  $\mathcal{V}$  whose domain of definition is a subset of  $\widehat{\mathbb{C}}$  and that fixes the marked point with multiplier one, its (suitably normalized) parabolic renormalization has at least structure  $\mathcal{V}'$ .

See Figure 16.

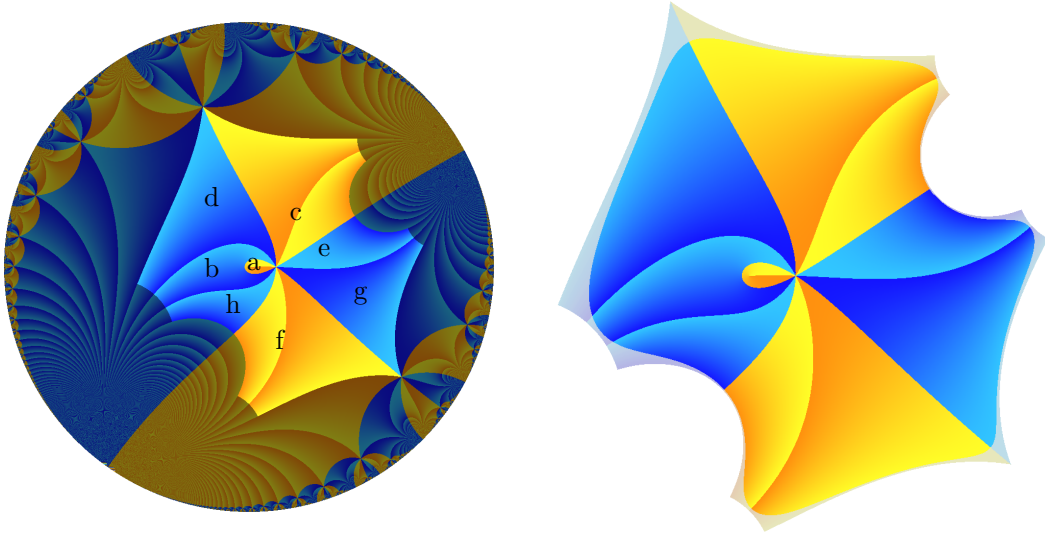


Figure 16: Left: The structure of  $B_3$  after first renormalization and its substructure  $\mathcal{V}'$  (the bright domain). Right: the comparison of  $\mathcal{V}$  and  $\mathcal{V}'$  ( $\mathcal{V}$  is a sub-structure of  $\mathcal{V}'$ ). The pictures are both accurately calculated. Although is hard to see, the boundaries of the light-toned domain and the color-saturated domain are disjoint.



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